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A GRAPHICAL METHOD OF SOLVING EMPIRICALLY RELATED DIMENSIONLESS PARAMETERS, WITH AN APPLICATION TO PIPE-LINE FLOW

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This paper presents a graphical method of solving certain problems involving empirical relationships between dimensionless parameters. Problems of this kind have heretofore been solved by the method of successive trials. The graphical method to be described is applicable to a large class of physical problems and is illustrated in the paper by an example taken from the field of fluid mechanics.

Statement of the Problem

Given a complete relation of unknown form

$$Q_1 = f(Q_2, Q_3 \text{ ---- } Q_N) \tag{1}$$

between the N physical quantities $Q_1, Q_2, Q_3, \ldots, Q_N$ involved in some physical situation. From Buckingham's π-theorem, the equation is reducible to the form

$$\pi_1 = \psi(\pi_2, \pi_3 \text{ ---- } \pi_{N-K}) \tag{2}$$

in which the arguments $\pi_1, \pi_2, \pi_3, \ldots, \pi_{N-K}$ are all the independent dimensionless products that can be formed by combining in any way the N physical quantities. The total number of such products is equal to N - K, where K is the number of fundamental dimensions needed to describe the N quantities. The form of the π's will be as follows:

$$\left.\begin{aligned}
\pi_1 &= Q_1^{\alpha_1} Q_2^{\beta_1} Q_3^{\gamma_1} \text{ ----- } Q_K^{\chi_1} Q_{K+1} \\
\pi_2 &= Q_1^{\alpha_2} Q_2^{\beta_2} Q_3^{\gamma_2} \text{ ---- } Q_K^{\chi_2} Q_{K+2} \\
\pi_3 &= Q_1^{\alpha_3} Q_2^{\beta_3} Q_3^{\gamma_3} \text{ ---- } Q_K^{\chi_3} Q_{K+3} \\
&\vdots \quad \vdots \quad \vdots \quad \vdots \quad \vdots \quad \vdots \\
\pi_{N-K} &= Q_1^{\alpha_{N-K}} Q_2^{\beta_{N-K}} Q_3^{\gamma_{N-K}} \text{ ---- } Q_K^{\chi_{N-K}} Q_N
\end{aligned}\right\} \tag{3}$$

in which the exponents $\alpha, \beta, \gamma, \ldots, \chi$ are pure numbers, some of which may be zero, whose values are fixed by the dimensions of the N quantities in such a way that the N - K π's shall be dimensionless.

The form of the function ψ cannot be determined solely from dimensional considerations. Indeed, dimensional analysis is simply a means of determining the least number of independent arguments of this function. However, the form of ψ may be determined experimentally and the equation $\pi_1 = \psi(\pi_2, \pi_3, \ldots, \pi_{N-K})$ may be plotted as a curve in N - K space.

Suppose that we desire to determine the value of some one of the N physical quantities, say

Q_1, which is common to both π_1 and π_2, but which either does not occur in the remaining π's or else occurs in them only in such a manner as to define the geometry of the physical system. If the remaining N - 1 quantities are given, the problem becomes simply to find a value P of the desired quantity Q_1 such that the point (π_1, π_2) evaluated at $Q_1 = P$ lies on the curve $\pi_1 = \psi(\pi_2, \pi_3, \ldots, \pi_{N-K})$. The usual method of solving this problem is by successive trials, assuming values of Q_1 until a value is found satisfying the above condition.

We shall now show how the method of successive trials may be replaced by a simple graphical method which readily yields the desired quantity Q_1.

Method of Solution

Eq (2) may be rewritten in the form

$$\log \pi_1 = \phi(\log \pi_2, \log \pi_3 \cdots, \log \pi_{N-K}) \tag{4}$$

Since the desired quantity Q_1 does not, in accordance with the above restriction, occur in the arguments $\log \pi_3, \ldots, \log \pi_{N-K}$ or else occurs in them only in such a manner as to define the geometry of the physical system, it will be seen that, for this analysis, we may consider only $\log \pi_1$ and $\log \pi_2$, treating all the other arguments as constants. For a given geometry, therefore, Eq (4) may be rewritten as

$$\log \pi_1 = \phi(\log \pi_2, \text{etc.}) \tag{5}$$

or

$$\log(Q_1^{\alpha_1} Q_2^{\beta_1} Q_3^{\gamma_1} \cdots Q_K^{\kappa_1} Q_{K+1}) = \phi\left[\log(Q_1^{\alpha_2} Q_2^{\beta_2} Q_3^{\gamma_2} \cdots Q_{K+2}), \text{etc.}\right]$$

or

$$\left[\log(Q_2^{\beta_1} Q_3^{\gamma_1} \cdots Q_K^{\kappa_1} Q_{K+1}) + \alpha_1 \log Q_1\right] = \phi\left\{\left[\log(Q_2^{\beta_2} Q_3^{\gamma_2} \cdots Q_K^{\kappa_2} Q_{K+2}) + \alpha_2 \log Q_1\right], \text{etc.}\right\}$$

But

$$\log(Q_2^{\beta_1} Q_3^{\gamma_1} \cdots Q_K^{\kappa_1} Q_{K+1}) = \log \pi_1 - \alpha_1 \log Q_1 \tag{6a}$$

and

$$\log(Q_2^{\beta_2} Q_3^{\gamma_2} \cdots Q_K^{\kappa_2} Q_{K+2}) = \log \pi_2 - \alpha_2 \log Q_1 \tag{6b}$$

Note that when Q_1 is unity, then $\alpha \log Q_1 = \alpha_2 \log Q_1 = 0$.

The point $\left[\log(Q_2^{\beta_1} Q_3^{\gamma_1} \cdots Q_K^{\kappa_1} Q_{K+1}),\ \log(Q_2^{\beta_2} Q_3^{\gamma_2} \cdots Q_K^{\kappa_2} Q_{K+2})\right]$

for which all necessary data are given, may therefore be plotted as the point

$$\left[\log(Q_1^{\alpha_1} Q_2^{\beta_1} Q_3^{\gamma_1} \cdots Q_K^{\kappa_1} Q_{K+1}),\ \log(Q_1^{\alpha_2} Q_2^{\beta_2} Q_3^{\gamma_2} \cdots Q_K^{\kappa_2} Q_{K+2})\right]_{Q_1 = 1}$$

in the N - K space containing the curve $\log \pi_1 = \phi$ (log π_2, etc.).

If Q_1 is assigned another value, say $(1 + \delta)$, a second point

$$\left[\log\left(Q_1^{\alpha_1} Q_2^{\beta_1} Q_3^{\gamma_1} \cdots Q_K^{\kappa_1} Q_{K+1}\right), \log\left(Q_1^{\alpha_2} Q_2^{\beta_2} Q_3^{\gamma_2} \cdots Q_K^{\kappa_2} Q_{K+2}\right)\right]_{Q_1=(1+\delta)}$$

may be plotted, and so on, until a point

$$\left[\log\left(Q_1^{\alpha_1} Q_2^{\beta_1} Q_3^{\gamma_1} \cdots Q_K^{\kappa_1} Q_{K+1}\right), \log\left(Q_1^{\alpha_2} Q_2^{\beta_2} Q_3^{\gamma_2} \cdots Q_K^{\kappa_2} Q_{K+2}\right)\right]_{Q=P}$$

is found lying on the curve $\log \pi_1 = \phi(\log \pi_2$, etc.) and satisfying the physical requirements of the problem. The locus of these points will be a straight line in log-log coordinates and the value $Q_1 = P$ will be the value sought.

However, it is not necessary that such a series of points actually be plotted, Since

$$\log\left(Q_2^{\beta_1} Q_3^{\gamma_1} \cdots Q_K^{\kappa_1} Q_{K+1}\right)$$

is displaced from $\log \pi_1$ by an amount $\Delta \pi_1 = - a_1 \log Q_1$ and since

$$\log\left(Q_2^{\beta_2} Q_3^{\gamma_2} \cdots Q_K^{\kappa_2} Q_{K+2}\right)$$

is displaced from $\log \pi_2$ by an amount $\Delta \pi_2 = - a_2 \log Q_2$, the slope m of the locus of points is

$$m = \frac{\Delta \pi_1}{\Delta \pi_2} = \frac{\alpha_1}{\alpha_2} \tag{7}$$

It follows that the intersection of the curve $\log \pi_1 = \phi(\log \pi_2$ etc.) and a line of slope $m = \frac{\alpha_1}{\alpha_2}$ drawn through the point

$$\left[\log\left(Q_1^{\alpha_1} Q_2^{\beta_1} Q_3^{\gamma_1} \cdots Q_K^{\kappa_1} Q_{K+1}\right), \log\left(Q_1^{\alpha_2} Q_2^{\beta_2} Q_3^{\gamma_2} \cdots Q_K^{\kappa_2} Q_{K+1}\right)\right]_{Q_1=1}$$

gives the values of $\log \pi_1$ and $\log \pi_2$ satisfying the physical situation and that the increments $\Delta \pi_1 = - a_1 \log Q_1$ and $\Delta \pi_2 = - a_2 \log Q_2$ may be scaled for $\log Q_1$ directly.

This result may be applied to a variety of physical problems. The following example, drawn from the field of fluid mechanics, is typical.

Application to Pipe-Line Flow

A problem which arises in designing pipe lines is that of determining the minimum diameter of pipe for a given length, flow rate, and allowable pressure drop.

The head loss h suffered by an incompressible fluid which is flowing with velocity V in a pipe of length L and diameter D is given by the Darcy-Weisbach equation

$$h = f \frac{L}{D} \frac{V^2}{2g} \tag{8}$$

in which f is the Darcy friction factor.

When this equation is derived by dimensional analysis it is found that the friction factor is a function of Reynolds' number, that is,

$$f = \psi\left(R, \frac{\epsilon}{D}\right) = \psi\left(\frac{DV}{\nu}, \frac{\epsilon}{D}\right) \tag{9}$$

where ν is the kinematic viscosity of the fluid and $\frac{\epsilon}{D}$ is the relative roughness of the pipe. The diameter D occurs in $\frac{\epsilon}{D}$ only in such a manner as to define the geometry of the pipe.

Figure 1 is a plot of the curve log f = ϕ (log R, $\frac{\epsilon}{D}$) after L. F. Moody (Trans., A.S.M.E., (1944), 671).

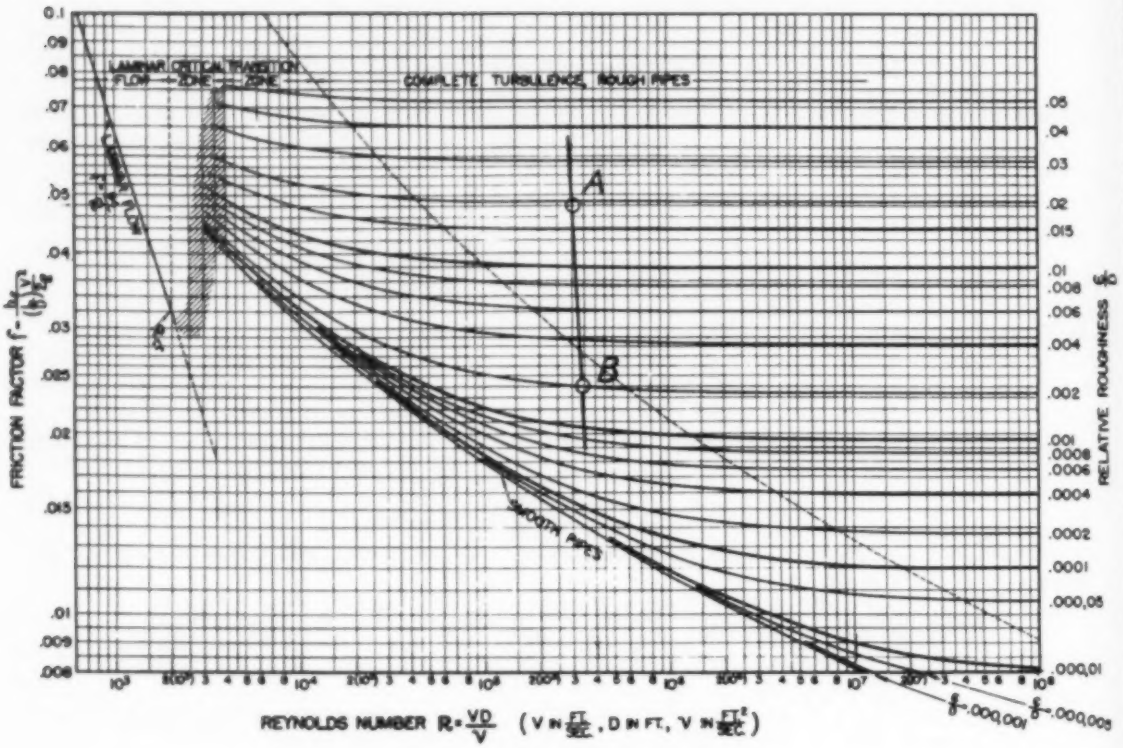

Figure 1

From Eq (8),

$$f = \frac{h}{\left[\frac{L}{D} \cdot \frac{V^2}{2g}\right]}$$

Then

$$\log\left(\frac{2gh\,D}{LV^2}\right) = \phi\left[\log\left(\frac{DV}{\nu}\right), \frac{\epsilon}{D}\right] \qquad (10)$$

But, from the equation of continuity for incompressible fluids, V = Q/A, where Q is the quantity of fluid flowing and A is the section area of the pipe. Hence,

$$\log\left(\frac{\pi^2 g h D^5}{8LQ^2}\right) = \phi\left[\log\left(\frac{4Q}{\pi\nu D}\right), \log\frac{\epsilon}{D}\right]$$

or

$$\left[\log\left(\frac{\pi^2 g h}{8LQ^2}\right) + 5\log D\right] = \phi\left\{\left[\log\left(\frac{4Q}{\pi\nu}\right) - \log D\right], \log\frac{\epsilon}{D}\right\}$$

from which it follows that

$$\log\left(\frac{\pi^2 g h}{8 L Q^2}\right) = \log f - 5 \log D,$$

$$\log\left(\frac{4Q}{\pi \nu}\right) = \log R + \log D$$

and $m = -5$

Taking a numerical case, suppose that one mile of concrete pipe is to carry 2.5 cubic feet of 70°F water per second. If the permissible head loss is 40 feet of water, what minimum diameter of pipe may be used?

Given that $\nu = 1.07 \times 10^{-5}$ square feet per second for water at 70°F. Solving for the point $(R, f)_{D=1}$, we obtain

$$\frac{4Q}{\pi \nu} = \frac{4 \times 2.5}{3.14 \times 1.07 \times 10^{-5}} = 2.97 \times 10^5$$

$$\frac{\pi^2 g h}{8 L Q^2} = \frac{(3.14)^2 \times 32.2 \times 40}{8 \times 5280 \times (2.5)^2} = 0.048$$

This point is plotted as point A in Figure 1. Through this point a line of slope $m = -5$ is drawn. The intersection of this line and the given function yields the values of R and f satisfying the physical situation. Taking a value of $\frac{\epsilon}{D} = 0.002$ as representative of the best grade of concrete pipe, we obtain from the graph (point B) the following values:

$$R = 3.40 \times 10^5$$

$$f = 0.024$$

The value of D may be scaled from the graph, taking care to use the correct scale factor. However, since

$$\frac{4Q}{\pi \nu} \div D = R$$

$$D = \frac{4Q}{\pi \nu} \div R = \frac{2.97 \times 10^5}{3.40 \times 10^5} = 0.875 \text{ FT.} = 10.5 \text{ IN.}$$

Similarly,

$$\frac{\pi^2 g h}{8 L Q^2} \times D^5 = f$$

$$D^5 = f \div \frac{\pi^2 g h}{8 L Q^2} = \frac{0.024}{0.048} = 0.5 \text{ FT}^5$$

$$D = 0.875 \text{ FT.} = 10.5 \text{ IN.}$$

Other applications occur in fluid metering, turbomachinery, heat transfer, elastic structures, and so forth.

OPERATIONS RESEARCH

An Address by Arthur A. Brown, given before the Industrial Mathematics Society, Detroit, Michigan, on May 16, 1952.

It is indeed a pleasure to me to be here tonight to address a meeting of the Industrial Mathematics Society. As a mathematician who has had occasion to work with industry, I feel that you have done an important thing in forming this society. Recognition of the value of industrial mathematics must benefit both the industrialist and the mathematician.

The program which your society has undertaken is most impressive. During the afternoon, as a guest of your officers, I have had a number of interviews with operating executives of business and industry. The cordiality and respect with which we were received is evidence of Detroit's appreciation of the work which all of you do.

However, I did not come here tonight to tell you how good your society is, but rather to discuss with you a kind of work which is allied to yours. Industrial mathematics, as I understand the term, refers to the use of mathematical techniques and the application of the mathematical type of thinking to the problems of design and control which arise in the normal process of production. A survey of the papers which have been read before your society shows that you deal with the design of gears, brake drums, engines, automobile frames, measuring instruments, torque transmitters, electrical distribution systems, systems for pumping fluids, and so on.

The kind of work reported on consists fundamentally of working out deductive conclusions from known (or partly known, partly assumed) laws of behavior of systems of inanimate matter. The peculiar value of mathematics in this process is that it affords us an economical and very flexible predictive tool. The applications of mathematics which I will discuss in the later parts of this talk are not essentially different in method; but they do differ in their object matter.

If I were addressing a group of executives from business or industry, it would be necessary for me to spend some time in discussing what is meant by the scientific method. We here tonight have an already formed idea of this method. I do not suppose that we would all agree on a definition; I have found from experience that a satisfactory definition of the scientific method is as hard to come by as is a satisfactory definition of operations research. Everyone agrees that the scientific method includes four phases: (1) observation, (2) inductive framing of hypotheses, (3) deductive formation of conclusions, and (4) comparison of these conclusions with reality, that is, experiment. In other words, it is generally agreed that science is founded on experiment, theory, prediction, and verification by experiment. But disagreement begins when we try to particularize the meaning of these generally accepted terms.

In order not to take up too much time tonight, therefore, I will take refuge in the knowledge that you are familiar with scientific method and in the assumption that since all of you know as well as I do what is meant by it, there is no need for me to say more precisely what I mean. I hope that no one will try to smoke me out of this refuge during the discussion period.

What we want to do now is to see whether there are areas of industry and business in which the scientific method can profitably be applied, and in which it is not now in use. The thesis of the proponents of operations research is that there are such areas, we believe that a

business enterprise, as an acting organization, is just as much a suitable object for scientific investigation as is a mechanical or electrical or chemical entity.

One word of caution is necessary here. A good deal of logical and intelligent thinking has gone into the study of management methods; and a collection of techniques, which is sometimes called "methods of scientific management" has grown up. It is not this body of techniques that we are concerned with. What we are after is a scientific understanding of the way the organization works and must work just because it is what it is. We want to know how it interacts with its environment, how it acts on itself, what its internal constraints are, and how they manifest themselves.

This last bit is rather high in viscosity, and I think you are entitled to ask whether it really means anything at all. Let us look at a rather simple example, a very simple one indeed. It is taken from the history of World War II rather than from industrial practice, because the wartime work in operations research is well documented, some is releasable for public study, and it is of general interest.

This example refers to the laws of operation of a squadron of the Royal Air Force, of the United Kingdom. During the late war, we were all short of manpower, and England was short indeed. She had a tremendous job to do in beating back the German submarine menace; her aircraft were employed in this as well as in the bombing of Germany, and, because of the high priority which the Bomber force enjoyed, the anti-submarine aircraft force was doubly pinched. The Operational Research Section of Coastal Command — which was that division of the RAF charged with anti-submarine duty — was asked to look into the problem of getting the maximum effort out of the existing Coastal Command equipment.

The job was undertaken by Dr. Cecil Gordon, a geneticist by peacetime training. He spent a much longer time at it than we can spend telling about it, and I will not try to put my narrative into the chronological order of his development of the results. I don't know what considerations led to his final formulation, and, because of security restrictions, I could not go into detail with you if I wished to. The story is written up in a book called Science at War, where you may see it if you like. I will begin my account with the outcome.

Suppose we divide up the life of the average aircraft of the squadron into four naturally occurring phases: flying, awaiting repair, undergoing repair, and awaiting flying duty. Let us denote the fraction of its life which the aircraft spends in these several phases by the symbols F, A, R, and W, respectively. Then we have the fundamental equation $F + A + R + W = 1$.

When the aircraft is flown, it is subject to wear and tear which must be made up by the repair force. Since we are dealing with the average aircraft, we neglect statistical fluctuations, and we say that each hour of flying gives rise to a certain number of hours of required repair time. This means that $R = kF$, where k denotes the repair time consequent on a unit of flying time. We might note in passing that some of the repair time is prescribed time — for example, the engines must be overhauled at stated intervals, by Air Force regulation — and part of it is due to breakdown or wear-out of parts. The revised equation, then, is

$$F(1 + k) + A + W = 1.$$

Once this fundamental equation is written down, and the constants evaluated by means of observation, the consequences of variations in policy and practice can be worked out by deductive methods. The mathematical portion of this example, that is, the deductive elaboration of consequences, is so very trivial that I hesitate to characterize it as mathematical work. In Gordon's detailed work, however, the mathematical theory — which allowed for fluctuation effects and the limiting effects of weather — was not as crude. And there are other examples from the military field, as you will see from a study of the fundamental text in Operations Research[1] — that is, Morse and Kimball — in which mathematics of a much higher order is

required. It is my belief that, if operations research strikes root in Detroit, those mathematicians associated with it will find reason to present many interesting papers to your society. Let us, however, return to our example.

At the time the study was undertaken, the squadrons were required by Headquarters policy to maintain an average of 75% of their aircraft in a condition fit to fly. That is, they were required to have

$$W = 1 - A - F(1 + k) \geq 0.75.$$

Clearly, this means that

$$F \leq (0.25 - A)/(1 + k),$$

which imposes a stringent maximum on the value of F. All of the norms and flying schedules were based on experience gained from operations subject to this constraint.

Gordon's proposal was that the policy requirement should be relaxed; that aircraft should be flown whenever they were ready and weather permitted; that the maintenance staff should no longer be rated with respect to the value of W which resulted, but rather by the ratio of F to total maintenance manpower.

His experimental observations led him to believe that a substantial increase in flying time could be obtained. The Command agreed to let one of its squadrons try the plan for a month. The trial showed that under the relaxed constraint, the natural value of W was 0.33, which meant

$$F = (0.67 - A)/(1 + k).$$

Not all of the apparent increase in F was realized, of course, because with increase in F the value of A increased as well. A 50% improvement was nevertheless secured at no cost in extra manpower.

One might say, perhaps, that the original policy requirement was nonsensical, and should never have been imposed. This would not be a correct judgement, however. The requirement that W be not less than 0.75 was imposed in peacetime, when the value of F was sharply limited by economy requirements, principally by limitations on fuel. In this case, the only free variable is A; and a large value of A means that the engineering and maintenance staff is falling down on its job. Thus, for peacetime conditions W is a proper control variable; it is not for wartime conditions.

The error lay in improper conversion from peacetime to wartime standards; its cause was the lack of appreciation of the fundamental equation.

Well, you may ask: Why did not the military staff write down the governing equation $F(1 + k) + A + W = 1$, and proceed to elaborate its consequences, without requiring the help of a scientist? I doubt if this question would occur to those of you who have had much experience with operating people; to those who have not, I can only say that apparently no operating officer can afford to stand back so far from his material. He is in the midst of the situation, doing his best to keep the machine running. He probably would not wish to take a distant look at it if he could.

Moreover, most operational people have not got an adequate symbolic equipment. They are not happy in the world of elementary algebra, not to speak of differential equations, matrix theory, and what have you. The scientific method simply cannot be applied by any and all intelligent people; except in certain rare cases, a thorough training is required to produce scientific habits of thought.

Special conditions are required for the application of the scientific method in any specific

area. In the field of Operations Research there are four essential requirements: the right kind of men must be chosen, they must be allowed to use the proper methods, in a suitable organizational matrix, on the proper sort of material. (You may wonder why I choose a 64-dollar word like matrix to denote the organizational framework. There are two reasons: principally that it begins with "m", and I need it for alliterative purposes; secondly that I cannot think of a more exact word for the concept I have in mind.)

The men must have scientific ability and the right kind of personal characteristics. They must be able to get along with all kinds of people in the organization, must know how to get information without offending the source and without thwarting the purpose of the experiment. They must have a feeling for practicality. These are not accidental characteristics; they are essential. It is as necessary that an operations research man be able to mix with operating people as that a bacteriologist should be able to prepare a culture, or a physicist able to read a voltmeter.

I will pass over the question of method rather briefly, as I have already said that we all know what the scientific method is. One aspect of Operations Research, however, is worth mentioning. O/R is research; it is an investigative science, not a branch of speculative philosophy. Its function is to arrive at the truth by inference from observation, not by logical operations on a set of a priori assumptions. It should be thought of as closer to the experimental sciences than to mathematics proper. The methods are those of experimental science generally; such a science has been aptly described as "an all-in wrestle with the facts." (I believe this remark is due to P. W. Bridgman.)

We come to the matrix. By this I mean the combination of (1) the correct position in the organizational chart, (2) the right kind of network of relationships to allow the group to have free access to the necessary information, and (3) a close relationship to the executive in charge of that portion of the organization which is being served by the group.

The group should be set up as part of the staff of the operating executive. Some method must be worked out to allow its members to study at close range any or all of the organization's activities. This means that a method must be worked out to generate and maintain close and friendly relationships with the parts of the organization whose work is being studied. In many of the military groups, this latter goal was achieved by the device of assigning members of the group as assistants to heads of subordinate activities — activities, that is, which were subordinate to the main command to which the central group was attached. Arrangements were made for the men assigned to correspond freely and directly with the central group; copies of all their reports to the subordinate commanders, for example, were routinely sent to the central group. The latter also retained the right to rotate its men among the various assignments. These arrangements, of course, might not be suitable in any given industrial organization, but some method of allowing the operations research men to circulate freely and unobtrusively is necessary.

Finally, it is wholly necessary that the head, at least, of the operations research group should be constantly aware of the problems facing the executive, and aware of the shifting degrees of importance and urgency which attach to these problems. This close relationship is obviously desirable; everybody wants to get into the act, and operations research men are like all others in this respect. But I say more than that — I say it is essential to the scientific validity of the group's findings. The close contact helps the operations research men to distinguish between what is intrinsic and what is arbitrary; and it enables them to secure prompt and effective testing of their hypotheses. Operations research which is carried out without contact with the executive is a contradiction in terms, like experimental physics carried out without contact with the real world.

Now for the last M — the material. What does the operations research group study? What problems does it undertake? In a general sense, the answer is: any problem of importance

to management and amendable to scientific study. In particular, however, the group is almost certain to find itself working on questions of metabolism — the relation of output to input — or, to put it in the bluntest way possible — questions of how to reduce costs. It will also study the relationship of the organization to its environment — which means prediction of sales, purchases, and so on — and will study the effect of variation of these on the well-being of the organization.

It is most unlikely, I should say from past experience, that an operations research group will supplant or compete with existing staff agencies set up to handle such activities as market research, design of methods and layouts, detailed studies of costs, and so on. What it will do is to complement these, providing additional information, general theoretical methods, norms or goals, etc. Every operations research group I have seen has acted as a stimulus to the growth, activity, and value of other staff agencies.

Let us look at a case in point. In a number of studies with which I have been associated during my connection with industrial O/R, the general problem was to determine the relationships between output and cost. The output might be measured by the dollar value of gross sales, let us say. We would note down this figure week after week, as obtained from some given division of the enterprise. We would also note the weekly total salaries paid to the people who made these sales. We had then a series of weekly observations (W_i, E_i). We would plot up a scatter diagram on a piece of coordinate paper, showing the general relationship of the E's to the W's. A preliminary study of this diagram would usually reveal a few anomalies; close inspection of the accounts in the appropriate weeks would then reveal the cause of the anomaly — either a bookkeeping distortion, a mischarge, or an exceptional circumstance. When the distortions and mischarges were corrected, and the data corresponding to the exceptional circumstances thrown out, we were always left with a scatter diagram of a very well-behaved type. In fact, all the scatter diagrams were linear, over a range of a whole order of magnitude in the variable W. (I am using the operations research man's concept of the order of magnitude; Morse and Kimball refer to it as the hemibel; one hemibel is a factor equal to $\sqrt{10}$, which is roughly equal to 3, which is roughly equal to e, the base of the system of natural logarithums.) The scatter was always very small — of the order of 5% of the mean value of the E's — and the correlation high. In other words, we always found a really linear relationship between sales and salaries.

This linear relationship was not homogeneous. The equation was of the form

$$E = q\,W + k, \tag{1}$$

where k was positive and far from being zero. The expenses E refer to the type of account which is normally regarded as being proportional to sales; no accounting methods would succeed in evaluating k by isolating expenses of one type or other; the E's were already purified, from the accounting point of view. (I have not got time to prove this to you: I must simply assure you that the value of k could be arrived at by inference only. Those of you who are interested in this aspect would find a good discussion in a little book by Philip Lyle.[2] Lyle is a director of Tate and Lyle, Ltd., the British sugar refiners; he is also a competent statistician.)

The usual equation for budgeting E would be

$$E = r\,W, \tag{2}$$

where $100\,r$ is the "percentage cost" of selling. Since r is determined by dividing average sales, the lines represented by these two equations intersect in the point $(\overline{W},\overline{E})$ corresponding to the average values of the two variables. At values of W below the average, the homogeneous equation (2) predicts values of E lower than those given by (1); at higher values of W, the reverse is the case. In other words, if the management is budgeting by means of (2), and the organization is operating according to (1), operating costs will look good when sales are above average, and bad when they are below.

If the true operating curve is known, then real deviations from normal operation can be detected, and progress can be made toward isolating the causes.

Once it is recognized that a pure moving expense category is in reality a composite, attention can be and will be focussed on the values of the constants q and k, and particularly on the way in which these values depend on the factors controllable by the organization. Such factors, for example, might be the layout of the selling department, the hiring and layoff policies, the compensation methods (commission, straight salary, overtime rates, etc.) predictability of sales, and other factors too numerous to mention.

I am sure that each of you will already have formed at least one hypothesis to account for the subsistence of equation (1), and it might be that in any given case your hypothesis would be correct. It is essential, however, for an operations research man to remember that his hypotheses must be not only plausible, but also in correspondence with experimental truth. For this reason, I am not going to attempt an explanation here tonight of the factors q and k, because I have not yet had the opportunity of exploring their nature in a real case. We were too busy making immediate use of the equation, and of the predictions we got from it, to manage an exploration of the fundamental questions. I hope to have an opportunity some day for further work along this line.

Studies of this kind always aided the other staff agencies to find real trouble spots. Their work, it turned out, was improved simply because it was directed to areas in which genuine savings could be made. Whenever a real deviation from a true operating curve is found, then either the deviation is in the direction of cost savings — in which case we have a good example before us, one worth studying for the positive lessons in it — or the deviation is in the direction of unnecessary expense — in which case we wish to find and eliminate the causes.

That is one elementary example of the kind of work which an operations research group would carry out. In the full scope of its charter, the group would work on many other problems, of a more sophisticated nature. To the extent that new concepts emerged from studies, it would expect that scientific method could contribute some thing of importance. So it has turned out in the military field, and so, I am convinced, it will turn out in business and industry.

The case for operations research cannot rest on analogies and isolated examples alone. Conviction that operations research is worthwhile must come, to those who have not tried it, from the belief that scientific method is the best method for studying those questions to which it is relevant. I am sure that all of us here tonight share this belief.

Let us consider, for a few moments, the practical questions involved in setting up an operations research group. Being a diagnostic and predictive agency, rather than an operating body, the group need not in general be large. Certainly it may begin in a small way. An executive who wishes to try out operations research need not feel that he must commit himself to an extensive program. Certain requirements must be fulfilled, but large investments, whether of money or of organizational commitments, are not among them. The executive must choose the right type of man, he must give him his full confidence and backing, and he must make serious and heavy demands on him — or in other words he must entrust him with important problems for study. This is not to say that earth-shaking results should be expected too early in the program. But the problems undertaken should be worth the time and energy expended by both the O/R man and the executive.

To sum up: Operations Research is the application of the scientific method, by an operations research group, to the study of the environment and operations of an organization, and to the determination of the basic laws of operation, in order to improve the effectiveness and efficiency of the organization in carrying out its basic purposes.

An operations research group should be made up of man with scientific training; they should

be allowed to employ scientific methods of research, in a proper framework, in the study of important organizational problems and phenomena.

Operations research can be begun in a small way in any organization which wishes to make a trial, and if properly managed, will return large dividends on the investment.

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AN INTRODUCTION TO HIGH-SPEED DIGITAL COMPUTATION

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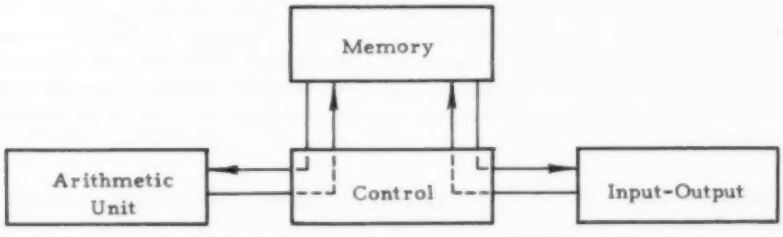
In the past few years enormous strides have been taken in the field of research and development of large-scale digital computing machinery. As a result of the advances in the field, the lay public has been inundated with a flood of claims of brainlike and humanlike qualities of these machines. To be sure, there is a widespread curiosity concerning machines which can add and multiply ten-decimal-digit numbers at rates up to a thousand times per second and perform logical, or elementary "thinking," operations. That curiosity is especially compelling to the engineer, the physicist, the mathematician, indeed to all scientists, for high-speed digital computation is not only becoming an important field in itself but is also becoming an important tool in all branches of scientific endeavor where computation is of use.

We shall be concerned here exclusively with the elements of organization and use of the large-scale, high-speed, electronic, digital computer. This type of computer utilizes electronic devices to the almost complete exclusion of mechanical devices and carries out a program of arithmetical and logical instructions in a self-sequenced, automatic manner. We shall discuss some of the salient characteristics of these computers from the standpoint of the user rather than the designer or builder, mentioning briefly only those engineering design aspects which seem especially important to the user. For a more complete account of much of the subject matter, the reader is referred to the articles and books listed in the Bibliography at the end of this paper.

General Organization of a Computer

The modern automatic computer can be regarded as consisting of four main components: the memory unit, control unit, arithmetic unit, and input-output mechanisms. The memory is that part of the machine in which numbers and instructions are stored. By means of the control unit, the component which interprets and carries out the instructions, numbers are transferred to the arithmetic unit, where they are processed (as in addition). The input-output equipment represents the machine's communication with the outside world; by means of it, numbers and instructions are placed in the memory and results obtained from the memory.



Some of the most important characteristics of a large-scale automatic digital computer are: (1) the number system used, (2) the memory system, and (3) the machine code. Each machine has a characteristic manner in which it stores and processes numbers depending on the number system used and the size of the largest number which can be handled by the machine.

The type of memory which a machine has is almost entirely an engineering design characteristic. However, the type of memory peculiar to a machine to a large extent determines its over-all speed, reliability, and storage capacity, and therefore the intended end use of the machine is a factor in determining the type of memory employed. The machine "code" is the manner in which a machine is given instructions and carries them out, in other words the machine "language."

Number Systems

A majority of the computing machine designers have turned to the binary number system as the basic machine number system. In this system each digit is either a "0" or a "1". Digits are coefficients of powers of 2 rather than powers of 10 as in the more familiar decimal system. The counterpart of the decimal point in this system is called the binary point. The position immediately to the left of that point is the units position, or the position of the zeroth power of 2, the second is the position of the first power of 2, and so forth. In general, the sequence of digits.

$$b_n b_{n-1} \ldots b_1 b_0 . b_{-1} b_{-2} \ldots b_{-(m+1)} b_{-m}$$

represents the number

$$b_n 2^n + \ldots + b_1 2 + b_0 + b_{-1} 2^{-1} + b_{-2} 2^{-2} + \ldots b_{-m} 2^{-m}$$

where each of the b's is either 0 or 1. For example, 1, 10, and 11, are the binary number representations of one, two, and three respectively. Thus we see that the addition of the binary numbers one and one produces a zero and a carry; that is $1 + 1 = 10$. One of the advantages of binary notation is that each digit can be simply represented in a machine by one of two electrical states. For example, the digit 1 can be represented by a pulse at a given time in a circuit while the absence of a pulse represents zero; or the digit can be represented by one of the two states of the so-called "flip-flop" vacuum-tube circuit. Furthermore, the rules of arithmetic in the binary system are especially simple.

The binary numbers, usually in lengths ranging from 30 to 50 digits (equivalent to 9 to 15 decimal digits), are stored in memory cells. The position of the binary point in these cells is, of course, important. Two methods are currently used in computers: the "floating point" and the "fixed point." The former means that there is not a definite position of the point and the position is changed depending on the problem. The fixed binary point is more commonly in use and means simply that numbers are always treated in the machine with the point in a standard position. For example, SEAC, the machine of the National Bureau of Standards, has its binary point fixed between the second and third binary digits. The total number of digits which can be stored in any given memory cell of SEAC is 45, including a sign digit. Therefore, the greatest number which can be stored is $4-2^{-42}$ (each digit in the cell is a 1).

The computing machine must also be designed to deal with the algebraic sign of numbers. This can be effected by reserving one digit or position in the memory cell as the sign position. In this case a 1 in the sign position implies that the number is negative and a 0 implies that the number is positive (or vice versa). An alternative method is to represent a negative number by the complement of its absolute value with respect to some standard positive number. The representation of negative numbers by complements will be explained further by the examples which follow.

Since the number of digits in the memory cells is finite, there must necessarily be some largest number which can be stored. "Overflow" occurs when an arithmetic process produces a number too large to be stored. It is obvious, for example, that in the case of SEAC if the number 2 is added to 3 an erroneous result will be found in the memory cell in which the correct sum, 5, was to be placed. This difficulty is obviated by the programmer who prepares the problem for the machine by the use of "scale factoring;" the programmer attaches to the various operands scale factors which adjust the numerical values in such a way as to

preclude the possibility of overflow. Some machines have specialized devices for the automatic detection and rectification of overflow.

Because a digital computer operates with discrete or quantized numbers the results of many operations cannot be recorded exactly. In SEAC, for example, any number to which a scale factor has already been applied which cannot be expressed as a sum of quantities of the form 2^n (where n is an integer in the range $-42 \leqq n \leqq 2$), cannot be represented exactly in the machine. Thus arises the familiar process of "rounding-off," the process of adding or leaving unaltered the least significant digit remaining after a number of digits are dropped. The digital computer often has provision for automatically rounding-off numbers in the processes of multiplication and division. With the binary number system the usual mechanization of round-off is simple: if the most significant digit dropped is 1, 1 is added to the remaining number in the least significant position; if the most significant digit dropped is 0, the remaining numbers are unchanged. For example, in rounding the number 101101 to 3 places, the 1 in the fourth position is added to 101 in the third position and the rounded number 110 is produced. A rounding-off process implies, in general, the production of a number which is in error. The accumulation of these errors can be serious in iterative processes which produce an answer after thousands of multiplications and divisions. The general topic of round-off errors is receiving considerable attention among researchers in the field of numerical analysis.

Let us fix our ideas with an examination of the number system used in the computer currently being built for the Oak Ridge National Laboratory. This machine has memory cells capable of storing 40 binary digits, and an extra digit is used in processing numbers. The total of 41 digits may be regarded as arranged as follows in the representation of a number:

$$p_1 p_0 . p_{-1} p_{-2} \cdots p_{-39}$$

Only numbers in the range $-1 \leqq N \leqq 1 - 2^{-39}$ are represented in the machine. A positive number is represented in the usual manner and a negative number as the complement of its absolute value with respect to 4. Therefore $p_1 = p_0 = 0$ in the case of a positive number and $p_1 = p_0 = 1$ in the case of a negative number. For example, the number 1/8 is represented by* 00.001 and the number -1/8 by 11.111. Thus the digits p_0 and p_1 are similar to sign digits. In the arithmetic processes of addition the machine completes the necessary steps as though all numbers are positive and in the range $0 \leqq N \leqq 4 - 2^{-39}$. As an example, the machine adds -1/4 to 3/8 in the following manner:

3/8	00.011
-1/4	11.110
1/8	00.001

The last carry-over is dropped and the correct representation of the sum 1/8 is obtained. Subtraction is carried out by first taking the complement of the absolute value of the subtrahend with respect to 4.

The digit p_1 is not, however, simply an extra sign digit. It can be used to detect whether overflow has occurred. If 3/8 is added to 3/4 the sum (9/8) is outside the acceptable range and overflow occurs. We illustrate here the normal addition process as the machine would do it:

3/8	00.011
3/4	00.110
	01.001

Here $p_1 \neq p_0$, signifying that overflow has taken place and the associated number is not correctly represented in the machine. The sensing of overflow is easily mechanized: the two

*As in examples to follow, we omit here the zeros following the least significant digit 1.

digits p_1 and p_0 are compared after each operation, and appropriate action taken if they are not equal. Provision will be made in the Oak Ridge computer for taking such action automatically.

The process of multiplication is simplified by the use of the binary number system. Multiplication is essentially an algorithm consisting of accumulating a sum by adding the multiplicand and shifting to the right if the first (least significant) digit is 1, or simply shifting to the right in the case of the binary digit 0, the process is then repeated for each successively more significant digit of the multiplier. For example, the process of multiplying 1/8 by 5/8 proceeds as follows:

1/8	00.001
5/8	00.101
add	00.001
shift	00.0001
shift	00.00001
add	00.001
	00.00101
shift	00.000101

and the correct answer, 5/64, is produced. Further examples of the processes of arithmetic processes are left to the reader. Rigorous proofs of the general correctness of the procedures in general are possible; we shall omit them here.

Memory Types

Of great importance in describing the over-all character of a computing machine is the type of memory which is built into it. Three types of memory are prominent in machines now in use or being planned for the immediate future: Magnetic drum, acoustic type (or mercury delay line), and electrostatic storage (or cathode-ray tube). The internal memory of modern digital computers consists of one of these three types; in addition most machines include external memory devices, such as magnetic tape, magnetic wire, or magnetic drum, to provide additional storage. All of the types of memory are characterized by elements which store binary digits in such a fashion as to make them readily available for transfer to other components of the machine or for interpretation by the machine as instructions. The memory is divided into a number of cells, each able to retain a single number in binary form. In discussing each type of memory in turn, we shall refer to the "access time" associated with it. The access time is that time required to obtain the number from storage and make it available for processing in the computer.

Magnetic drum storage is very popular because of its low cost per digit stored and because of its reliability. It consists of a cylinder whose surface is coated with a substance of favorable magnetic properties. During operation the drum rotates at constant speed under "write" and "read" heads which enter numbers onto the drum, and detect the stored numbers, respectively. The surface of the drum is divided into circumferential tracks, and each track in turn is divided into a number of elements, each of which is magnetized for the storage of a binary digit. The digit 1 is represented by magnetization of one polarity, the digit 0 is represented by magnetization of the opposite polarity. As many as 100 binary digits can be stored per inch of circumference. Access time for this type of memory depends on the speed of rotation of the drum and on the position of the digits to be read. If the position of the digits to be read is such that at the time of the read command the digits have just passed under the read head then a complete rotation will be required to bring them into position.

The drum of the MIDAC computer being built for and by the University of Michigan is 8.5 inches in diameter, 14 inches long, and rotates at 3450 rpm. It can store 276,480 binary digits arranged into 6,144 memory cells, each 45 digits in length. The maximum access time is about 17 milliseconds (1 millesecond = 0.001 seconds), and the average is about 9 milliseconds.

Such access times are representative for magnetic drum installations. Magnetic drum memory is often used as auxiliary memory and installed to be used in conjunction with an internal memory of lower access time. Such is the case of the MIDAC computer, in which the drum is used in conjunction with the faster acoustic memory.

The acoustic-type memory utilizes mercury delay lines to store numbers. The binary digit one is stored as a sound wave or pulse occupying a particular position at a given time in a column of mercury contained in a glass tube. The absence of a pulse corresponds to zero and a pulse corresponds to one. These pulses are used throughout the computer. Because of the relatively slow speed of propagation of sound in mercury a large number of pulses can be stored in a tube of moderate length. Quartz crystals at the extremities serve to propagate the pulses at one end and to receive them at the other. Provision is made for the recirculation of these pulses through the tube. Read-in and read-out is effected by gates which, at appropriate times in the circulation cycle, allow pulses to be produced and erased, or sense the presence or absence of a pulse. Delay lines presently in use on the SEAC and MIDAC computers allow for the storage of 384 binary digits. With both these computers the digits are grouped into cells of 48 digits each for computer operation, so that each delay line stores 8 cells. A total of 64 such lines gives each of these computers 512 memory cells of the acoustic type. The maximum access time for the delay line is 384 microseconds (1 microsecond = 0.000001 seconds), and the average is 192 microseconds, making it considerably faster than the magnetic drum.

The third type of memory, the electrostatic type, utilizes a cathode ray tube with an electron gun which changes the charge of the target, to effect a correspondence, for a given point on the target, between the charge and the binary digit zero or one. The target of each tube is in the form of a square array of, say 16 by 16, making possible the storage of 256 binary digits. The number of digits which can be stored depends on the particular design of the tube, and some success has been found with tubes storing as many as 1024 digits (a 32 by 32 array). Read and write circuits are usually so arranged that all of the binary digits corresponding to a memory cell can be stored or read simultaneously (in the jargon of computer design, "in parallel"). Therefore if 40 tubes were provided each with a 16 by 16 arrary, the memory would have a capacity of 256 40-digit memory cells. A given cell would then correspond to a certain position in the square arrary of each of the tubes. The electrostatic storage used in this parallel fashion means that access time is very small, usually between 10 and 30 microseconds.

The magnetic drum memory and the acoustic memory are usually employed in machines which operate in "serial" fashion (the digits of a cell read or written sequentially). Serial operation and relatively slow access time imply that machines using these types of memory are slower than those using the electrostatic type. The acoustic memory machines are, however, faster than those employing the magnetic drum type as the basic, internal memory. Representative speeds for addition in machines using the three types of memory are 30, 1, and .05 milliseconds for magnetic drum, acoustic and electrostatic, respectively. On the average, multiplication takes about 5 times as long as addition. The reliability of a memory installation depends, to a great degree, on its particular design environment. In general, the magnetic drum type is most reliable and requires the least maintenance, the acoustic type is less reliable, and the electrostatic type is the least reliable.

Machine Codes

Machine codes can probably best be described by referring to the code of a specific machine and regarding other types of codes as variations of it. We shall briefly describe the code of MIDAC, the University of Michigan computer.

MIDAC has 512 acoustic memory cells for a fast, internal memory, and the magnetic drum provides an auxiliary memory of 6,144 cells. Each memory cell consists of 45 binary digits,

including a sign digit. The name of each cell is a number called its "address." The contents of each cell is interpreted by the machine either as an instruction or as a number to be involved in an arithmetic or logical operation. The general term for information stored in a cell is "word." Numbers are stored with the binary point on the extreme left, that is immediately to the left of the most significant digit; hence, all numbers are in the range $-1 < N < 1$. The machine in most cases carries out the command or instruction stored in a cell by performing operations with two numbers stored in other cells, places the result in a fourth cell, and proceeds to a fifth for the next instruction. The digits in a cell are divided into several groups to provide the necessary information: the addresses of the two operands, the address of the cell into which the result is to be placed, a group of control digits, and the code number corresponding to the particular operation (for example, addition) to be performed. We shall call the addresses α, β, and γ, respectively. The control digits make the code more flexible by making the addresses relative to certain counters. The details of their use are not essential to our discussion. Thus we see that an instruction is placed in the machine as a 45-digit word, and is interpreted by the machine as follows: the sign digit is usually ignored; beginning with the most significant part of the number, twelve digits are allowed for each of the three addresses α, β, and γ in that order. The next four digits are the control digits, and the last four specify the operation code.

MIDAC has a total of 16 operations, which include arithmetic operations, logical or decision operations, and control functions such as data transfer to and from the input-output units and to and from the magnetic drum. The operations used in the example below, together with the symbols by which we shall refer to them, are as follows:

A	addition
S	subtraction
M	multiplication with round-off
D	division, unrounded
K	absolute value comparison.

We adopt the following convention: a primed quantity refers to the number in the cell designated by the quantity. Thus, α represents an address, and α' the number stored in cell α.

The division operation is

$$\beta' \div \alpha', \text{ the result placed in } \gamma.$$

The absolute value comparison instruction is described as follows:

$$\text{if } \alpha' \geq \beta' \text{ next instruction is in next cell;}$$
$$\text{if } \alpha' < \beta' \text{ next instruction is in } \gamma.$$

This instruction gives the machine the ability to make simple decisions and to select between two branches of control in carrying out a program of instructions. The machine processes instructions in consecutively numbered memory cells until a conditional order of this type is encountered. The manner in which the machine carries out the remaining operations is obvious from the explanation in the above paragraph. The other operations which MIDAC can perform will not be described here. They involve such processes as, for example, multiplication unrounded, and certain data transfer operations.

A device often used by the coder in preparing a program for a machine is called the "inductive loop." Indeed, the inductive loop illustrates one of the most powerful features of the high-speed computer: the repeated application of a set of instructions, with the iteration process stopping automatically at the proper time and the control returning to a specified memory cell. The general flow diagram for the inductive loop is seen in Figure 1. The loop is entered upon the completion of a number of orders which assure the processing of the first step upon entering. The first component in the loop is the "operation" which, in general, carries out the n-th step. The "decision" is usually a comparison operation which, in effect, decides whether the n-th step shall be the last step, and, as a result of the comparison,

transfers control to "substitution" or out of the loop. The component marked "substitution" carries out instructions necessary to insure that the operation component will now complete the (n + 1)-th step. The inductive loop is akin to the mathematical process called "mathematical induction" in which the (n + 1)-th step is defined upon the completion of the n-th step. The inductive loop is usually a small part of a large program of instructions. In general a small program which is used as a component in this sense is referred to as a "subroutine."

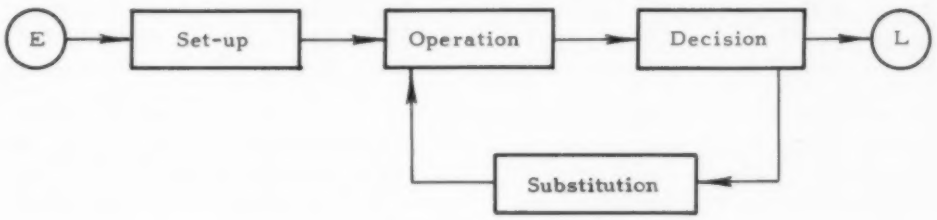


Figure 1.

Let us discuss a specific example of an inductive loop using the MIDAC code. We shall suppose that as part of a main program it is desired to find the cosine of the angle θ (in radians) to the full accuracy of the machine. The set of instructions for the cosine will then be a subroutine. In general, if x is in radians then we may expand cos x in an infinite power series about x = 0.

$$\cos x = 1 - \frac{x^2}{2} + \frac{x^4}{4!} - \cdots$$

We shall designate the n-th term of this series by a_n so that

$$a_n = \frac{(-1)^n x^{2n}}{(2n)!} = \frac{-x^2}{(2n)(2n-1)} a_{n-1}$$

and $a_0 = 1$. The sum after n terms, C_n, can be defined in a recursive manner also:

$$C_n = \sum_{i=0}^{n} a_i = C_{n-1} + a_n.$$

The counterparts of the general inductive loop of figure 1 are given in figure 2 for this specific case. The operation consists of obtaining the term a_n from the term a_{n-1} by multiplication by the factor as indicated and by forming the n-th partial sum C_n by adding a_n to C_{n-1}. The number of terms required for a given accuracy depends on the magnitude of θ. But regardless of θ, we wish to obtain successively more accurate partial sums, C_n, until the term a_n is so small that it cannot be recorded in the machine; i.e., until a_n is recorded in the machine as zero. The next step is a comparison of $|a_n|$ with 2^{-44} (the smallest number which can be stored in the machine) so that if $|a_n| < 2^{-44}$ the control eaves the loop and if $|a_n| \geq 2^{-44}$ the instructions are carried out to obtain the (n + 1)th partial sum. The arrows in the figure simply refer to the transfer of numbers within the machine. For example, $2n + 1 \rightarrow 2n-1$ means that the number 2n + 1 is placed in a certain cell which now nominally contains 2n-1. The set-up orders ensure that the proper quantities a_0 and C_0 are in the proper cells which will contain a_n and C_n, and that the first quantities obtained will be a_1 and C_1.

We complete the example by writing down the orders as a machine programmer would. We shall suppose that the instructions are to be stored in consecutive cells beginning with zero. We shall not suppose any particular location of the various constants and quantities computed. Rather, we shall use the symbol [x] to mean "the address of the memory cell which does contain or is reserved to contain the number x." We suppose that the angle θ is stored with the scale factor 2^{-2}. The symbol [L] designates the address of the first instruction after the subroutine, and [T] is the address of a certain cell which is used to store quantities temporarily. The code is as follows:

Memory Call Address	α	β	γ	Operation	Explanation
0	$[\theta.2^{-2}]$	$[\theta.2^{-2}]$	[T]	M	$\theta^2.2^{-4} \rightarrow -x^2.2^{-4}$
1	[0]	[T]	$[-x^2.2^{-4}]$	S	$-\theta^2 2^{-4} \rightarrow -x^2.2^{-4}$
2	$[1\cdot2^{-2}]$	[0]	$[C_{n-1}\cdot2^{-2}]$	A	$1\cdot2^{-2} \rightarrow C_n\cdot2^{-2}$
3	$[1\cdot2^{-2}]$	[0]	$[a_{n-1}\cdot2^{-2}]$	A	$1\cdot2^{-2} \rightarrow a_n\cdot2^{-2}$
4	$[1\cdot2^{-10}]$	[0]	$[(2n-1)2^{-10}]$	A	$1\cdot2^{-10} \rightarrow (2n-1)2^{-10}$
5	$[2\cdot2^{-10}]$	[0]	$[(2n)2^{-10}]$	A	$2\cdot2^{-10} \rightarrow (2n)2^{-10}$
6	$[(2n-1)2^{-10}]$	$[(2n)\cdot2^{-10}]$	[T]	M	$(2n)\ (2n-1)2^{-20}$
7	[T]	$[2^{-22}]$	[T]	D	$2^{-2}/(2n)\ (2n-1)$
8	[T]	$[a_{n-1}\cdot2^{-2}]$	[T]	M	$a_{n-1}\cdot2^{-4}/(2n)(2n-1)$
9	[T]	$[-x^2.2^{-4}]$	[T]	M	$a_n\cdot2^{-8}$
10	$[2^{-6}]$	[T]	$[a_{n-1}\cdot2^{-2}]$	D	$a_n\cdot2^{-2}$
11	$[a_{n-1}\cdot2^{-2}]$	$[C_{n-1}\cdot2^{-2}]$	$[C_{n-1}\cdot2^{-2}]$	A	$C_n\cdot2^{-2}$
12	$[a_{n-1}\cdot2^{-2}]$	$[2^{-44}]$	[L]	K	$0 = \lvert a_n \rvert$?
13	$[(2n-1)2^{-10}]$	$[2\cdot2^{-10}]$	$[(2n-1)\cdot2^{-10}]$	A	$(2n+1)2^{-10} \rightarrow (2n-1)2^{-10}$
14	$[(2n)2^{-10}]$	$[2\cdot2^{-10}]$	$[(2n)2^{-10}]$	A	$(2n+2)2^{-10} \rightarrow (2n)2^{-10}$
15	[0)	$[2\cdot2^{-10}]$	6	K	go to cell 6

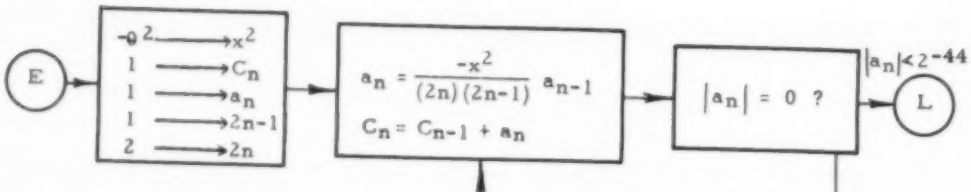


Figure 2.

The set-up orders, stored in cells 0 through 5, ensure that the quantities are in the correct place. They may be necessary because the cells marked $[-x^2.2^{-4}]$, $[C_{n-1}\cdot2^{-2}]$, etc., might have been used previously to store other numbers. Instructions 6 through 14 are processed iteratively, and the unconditional transfer order 15 always returns control to instruction 6. The iteration stops when the comparison order 12 transfers control out of the loop. Scale factoring is used here to prevent overflow. Recalling that numbers in the machine must be less, in absolute value, than $1-2^{-44}$ we note that overflow does not occur as long as $|\theta| \leq 4-2^{-42}$. The range of θ for which this subroutine is valid is thus defined. After the control leaves the loop the quantity cos θ is found in the cell marked $[C_{n-1}\cdot2^{-2}]$ with the scale factor 2^{-2} and will be correct to approximately 10 decimals.

The type of code employed in the MIDAC is called "three-address," because three addresses are involved in each instruction. The SEAC employs the "four-address" code, with the addresses of two operands, the result, and the next instruction all being specified. Another type of code is the so-called "single-address." With the single-address code, numbers are transferred to an accumulator in which the normal arithmetic and logical processes are carried out. In the single-address system, adding two numbers and placing the sum in the memory requires two instructions provided one of the numbers is initially in the accumulator. One instruction causes the second number to be transferred to the accumulator and the sum to be formed, the second instruction causes the sum to be placed in the memory. However, in this case the smaller amount of information required for each instruction permits two instructions to be stored in one memory cell.

One of the commonest methods of loading a machine's memory with a program of instructions is with the use of standard Teletype paper tape. With MIDAC, for example, each line of coding (an instruction or a constant) may be compiled into an eleven digit number in the "hexadecimal" system, that is, a number to the base 16. Since a hexadecimal digit can be represented by 4 binary digits, each digit of the code can be recorded on the paper tape by the use of 4 levels across the width of the tape, each level being punched or not punched according to whether a 0 or 1 is to be recorded. It is possible, however, to insert instructions in decimal form. Other machines have comparable methods of placing the code on paper tape. Input mechanisms vary a great deal from one machine to another. Mechanism in use today involve the reading of paper tape into the computer by modified Teletype equipment, or the faster photo-electric reader. In some cases the information on paper tape is transferred to magnetic wire for greater speed in reading into the machine. Output mechanisms also vary a great deal, the commonest is typing of information by modified Teletype equipment or Flexowriter equipment.

Uses of Computers

One of the great features of machines of this type is the high speeds with which they operate. MIDAC, for example, can add and subtract two 44-binary-digit numbers (each equivalent to slightly more than 11-decimal-digits) in about 1.2 milliseconds. The average time for multiplication or division of two numbers is about 3.0 milliseconds. These figures include the time for obtaining the operands from the memory, completing the operation, and placing the result in the memory. This speed of operation would enable the computation of the cosine of an angle to 10-decimal accuracy with the program written above in an average of about 0.1 seconds, where 7 terms of the series would be formed and summed. The speed of MIDAC is representative of computer speeds in general. However, as we have indicated above, some computers operate at greater speeds. Another computer at the University of Michigan operates about 30 times as fast as MIDAC.

Another principal feature of the modern computer is the elementary logical operations, sometimes called "thinking" process, which it can carry out. For example, programs have been contrived which will enable a machine to evaluate and choose moves corresponding to various play situations in the game of checkers.* Thus the machine is able to "play" checkers.

These two features make digital computers desirable for use in diverse fields of scientific endeavor. Practically any problem which is capable of a mathematical formulation can be successfully attacked. In particular some of the problems for which the machines can be used are the following:

*C. S. Strachey, National Research Development Corporation, London, England, reported on a machine program for playing checkers at the meeting of the Association for Computing Machinery of September 8 and 9, 1952. The paper "Logical or Non Mathematical Programmes" will be published in the proceedings of that meeting.

1. Solution of large sets of linear simultaneous equations, the inversion of high order matrices, and the determination of eigenvalues and eigenvectors of matrices.
2. Solution of systems of ordinary or partial differential equations and boundary value problems.
3. Determination of roots of algebraic or transcendental equations.
4. Solution of the minimization and maximization problems of the Calculus of Variations.
5. Computation of the various coefficients and functions in the statistical treatment of large volumes of data.

These mathematical problems are involved in the problems of such fields as engineering, physics, and econometrics. Solutions of suitable sets of equations, together with the use of the logical or conditional instructions, make these machines very valuable in the simulation of large and complex systems which are thereby tested by the building of a mathematical model.

Because high-speed computation is a comparatively new field, major improvements in machine design and use are forthcoming. In particular, considerable attention is being given to the problem of machine reliability. Also, the basic expense of machines is being reduced. At the present time, many machines compute more rapidly than is warranted by the speed with which instructions can be read in and answers read out. The ultra-high-speed Michigan Computer mentioned above is a special-purpose machine which is not limited in this way. Higher speed input-output devices are being designed, built and installed into existing machines and planned for incorporation into future machines. Meanwhile, research continues in improving techniques for the use of the machines. For example, considerable progress has been made toward the simplification of machine codes and operation. Also, the development of the machines has emphasized the necessity of developing techniques in numerical analysis. The new techniques apply to the solution of the mathematical problem mentioned above as well as to the error enalysis of those solutions.

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ENERGY APPROACH TO STRESS PROBLEMS

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Introduction

The following paper is derived from lectures presented to the Energy Study Group of The Industrial Mathematics Society. It is here intended to show the application of the energy method to the solution of problems in deflection as well as to problems in statically indeterminate structures. The basis of the method shown here is the Castigliano Theorem.

In this paper, all of the usual assumptions regarding elastic materials are presumed, namely that stress is proportional to strain, the material is isotropic, the principle of superposition holds, etc. However, the application of this method can be extended to the use of non-isotropic materials such as reinforced concrete structures.

Energy Stored in Member

Straight Member with Axial Load

Constant Load

Constant Section

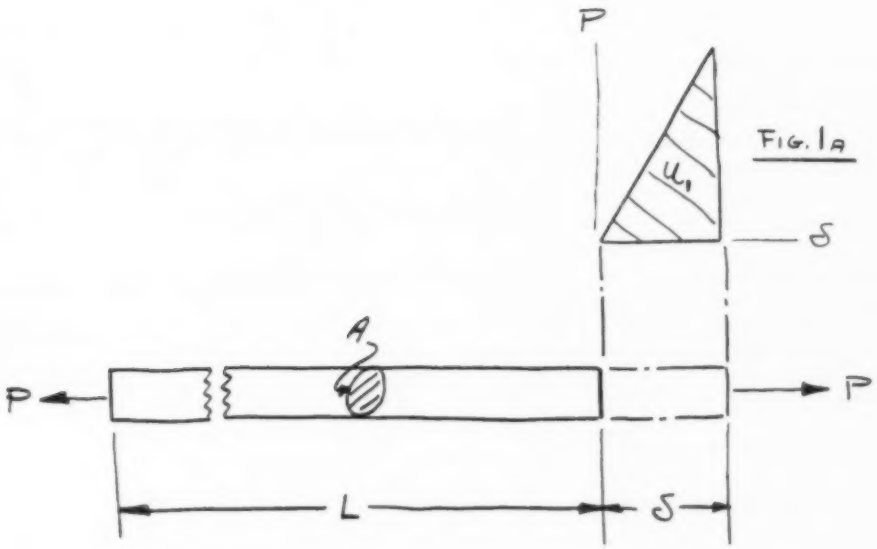

FIG. 1A

FIG. 1B

Figure 1

L = length of member = in.
A = area of cross section of member = $in.^2$
P = total axial load on member = lbs.
δ = total elongation of member = in.

S = stress in member = $\frac{P}{A} = \frac{lbs.}{in.^2}$

e = strain in member = $\frac{\delta}{L} = \frac{in.}{in.}$

E = modulus of elasticity = $\frac{S}{e} = \frac{PL}{A\delta} = \frac{lbs.}{in.^2}$

then $\delta = \frac{PL}{EA}$

U_1 = energy stored in member = lb. in.

$$U_1 = \frac{P\delta}{2} = \frac{P}{2}\,\frac{PL}{EA} = \frac{P^2L}{2EA} \qquad (1)$$

Straight Member with Shear Load

Constant Load

Constant Section

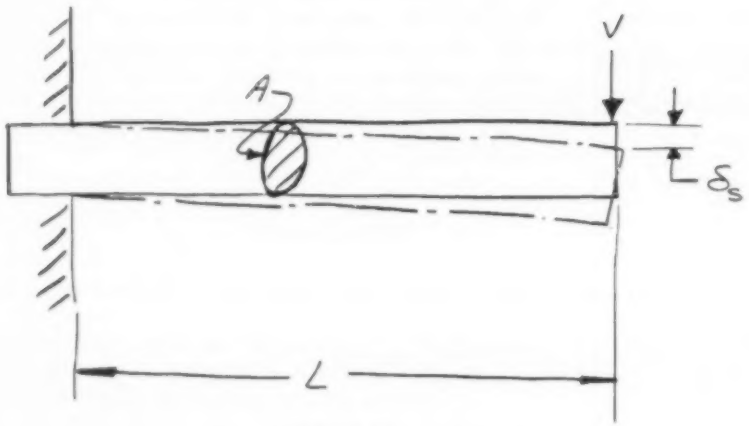

Figure 2

L = length of member = in.
A = area of cross section of member = $in.^2$
V = total shear load on member = lbs.
δ_S = total shear deflection of member = in.

S_S = shear stress in member = $\frac{V}{A} = \frac{lbs.}{in.^2}$

γ = Shear strain in member = $\frac{\delta_S}{L} = \frac{in.}{in.}$

G = modulus of elasticity in shear = $\frac{S_S}{\gamma} = \frac{VL}{A\delta_S} = \frac{lbs.}{in.^2}$

then $\delta_S = \frac{VL}{GA}$

U_2 = energy stored in bar due to shear = lb. in.

$$U_2 = \frac{V\delta_S}{2} = \frac{V}{2} \cdot \frac{VL}{GA} = \frac{V^2L}{2GA} \qquad (2)$$

Straight Member with Bending Moment

Constant Moment

Constant Section

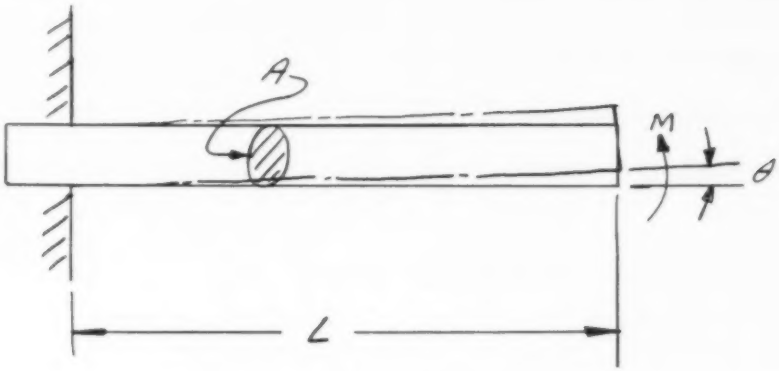

Figure 3

L = length of member = in.
A = area of cross section of member = $in.^2$
I = moment of inertia of section = $in.^4$
M = total applied moment = lb. in.
θ = total angular deflection = radians
c = extreme fibre distance = in.

S = max. bending stress in member = $\frac{MC}{I} = \frac{lbs.}{in.^2}$

e = max. strain in member = $\frac{\theta c}{L} = \frac{in.}{in.}$

E = modulus of elasticity = $\frac{S}{e} = \frac{ML}{I\theta} = \frac{lbs.}{in.^2}$

then $\theta = \frac{ML}{EI}$

U_3 = energy stored in member due to bending = lb. in.

$$U_3 = \frac{M\theta}{2} = \frac{M}{2} \cdot \frac{ML}{EI} = \frac{M^2L}{2EI} \qquad (3)$$

Straight Member with Torque Load

Constant Torque

Constant Section (Circular Section)

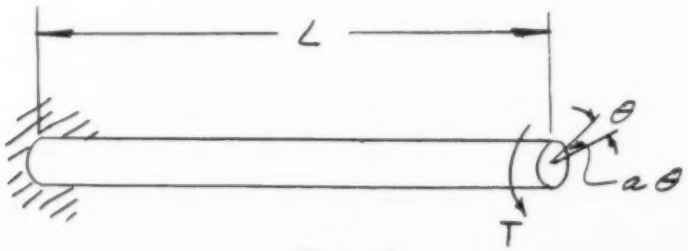

Figure 4

L = length of member = in.
A = area of cross section of member = $in.^2$
J = polar moment of inertia of section = $in.^4$
T = total applied torque = lb. in.
θ = total angular deflection = radians
a = radius of section

S_S = max. shear stress in member = $\frac{Ta}{J} = \frac{\text{lbs.}}{\text{in.}^2}$

γ = max. shear strain in member = $\frac{a\theta}{L} = \frac{\text{in.}}{\text{in.}}$

G = modulus of elasticity in shear = $\frac{S_S}{\gamma} = \frac{TL}{J\theta} = \frac{\text{lbs.}}{\text{in.}^2}$

then $\theta = \frac{TL}{GJ}$

U_4 = energy stored in member due to torsion = lb. in.

$$U_4 = \frac{1}{2} T \theta = \frac{1}{2} T \cdot \frac{TL}{GJ} = \frac{T^2 L}{2GJ} \qquad (4)$$

Total Energy Straight Member

Constant Load

Constant Section

Let U = total energy stored in member = lb. in.

Then in general

$U = U_1 + U_2 + U_3 + U_4$

= Energy due to axial load (1)
+ Energy due to shear load (2)
+ Energy due to bending (3)
+ Energy due to torsion (4)

$$\text{or } U = \frac{P^2 L}{2EA} + \frac{V^2 L}{2GA} + \frac{M^2 L}{2EI} + \frac{T^2 L}{2GJ} \qquad (5)$$

Straight Member with Axial Load

Varying Load

Varying Section

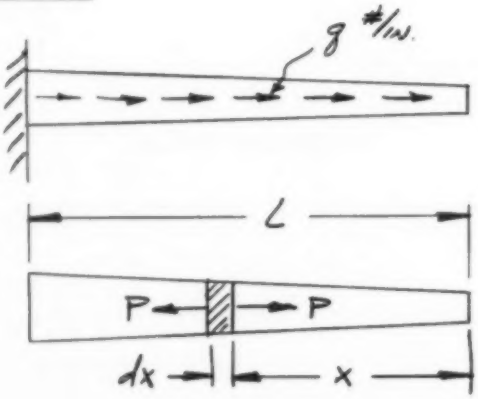

Figure 5

From Eq. 1 letting L = dx and remembering that P and A are variables and letting d U, = energy stored in increment of Fig. 5.

$$d\,U_1 = \frac{P^2 dx}{2\,EA} \tag{1a}$$

$$\text{then } U_1 = \int_0^L \frac{P^2 dx}{2\,EA} \tag{1b}$$

Straight Member with Shear Load

Varying Shear

Varying Section

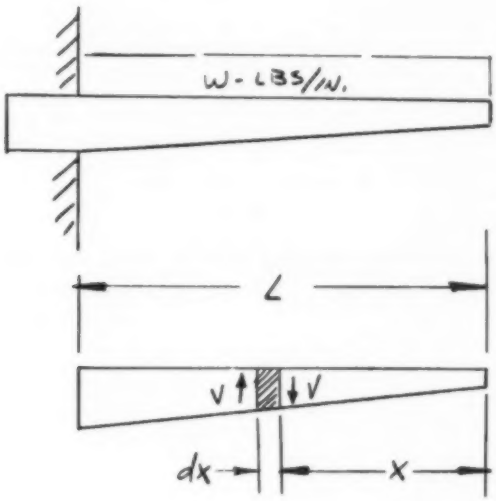

Figure 6

From Eq. 2 letting L = dx and remembering that V and A are variables and letting d U_2 = energy stored in increment of Fig. 6.

$$d\,U_2 = \frac{V^2 dx}{2\,GA} \tag{2a}$$

$$\text{then } U_2 = \int_0^L \frac{V^2 dx}{2\,GA} \tag{2b}$$

Straight Member with Bending Moment

Varying Moment

Varying Section

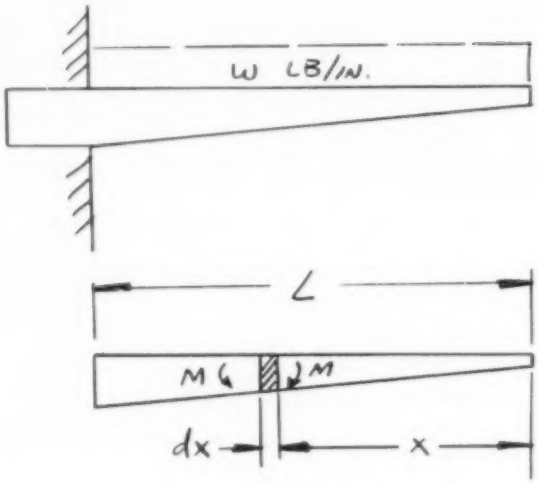

Figure 7

From Eq. 3 letting L = dx and remembering that M and I are variables and letting d U_3 = energy stored in increment of Fig. 7.

$$d\,U_3 = \frac{M^2\,dx}{2\,EI} \tag{3a}$$

$$\text{then } U_3 = \int_0^L \frac{M^2 dx}{2\,EI} \tag{3b}$$

Straight Member with Torque Load

Varying Torque

Varying Section

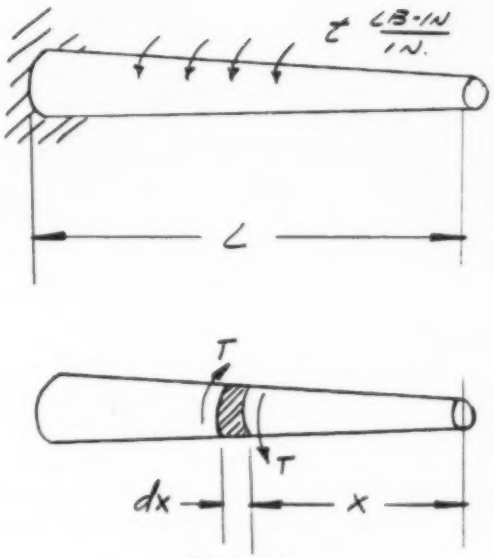

Figure 8

From Eq. 4 letting L = dx and remembering that T and J are variables and letting d U_4 = energy stored in increment of Fig. 8.

$$d U_4 = \frac{T^2 dx}{2 GJ} \tag{4a}$$

$$\text{then } U_4 = \int_0^L \frac{T^2 dx}{2 GJ} \tag{4b}$$

Total Energy Straight Member

Varying Load

Varying Section

Let d U = total energy stored in increment = lb. in.

Then in general

$$d U = d U_1 + d U_2 + d U_3 + d U_4 \tag{5a}$$

$$= \frac{P^2 dx}{2 EA} + \frac{V^2 dx}{2 GA} + \frac{M^2 dx}{2 EI} + \frac{T^2 dx}{2 GJ}$$

Then U = total energy stored in member

$$U = \int_0^L \frac{P^2 dx}{2 EA} + \int_0^L \frac{V^2 dx}{2 GA} + \int_0^L \frac{M^2 dx}{2 EI} + \int_0^L \frac{T^2 dx}{2 GJ} \tag{5b}$$

In the above equations P, V, M, T, A, I, and J may be variables.

Curved Member

Varying Loads

Varying Section

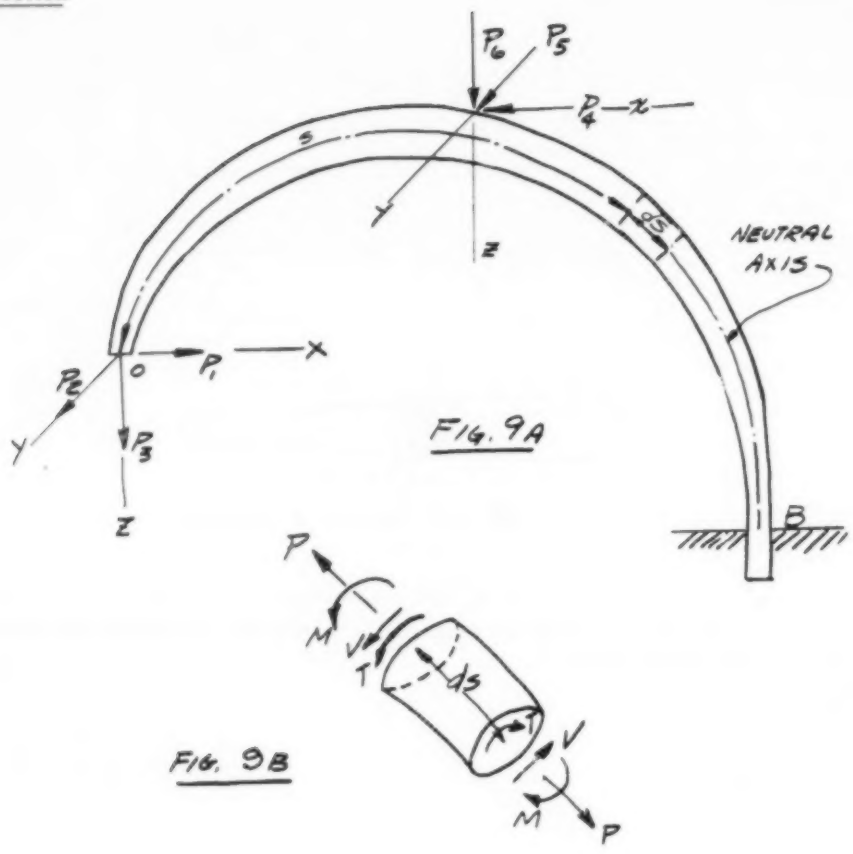

FIG. 9A

FIG. 9B

Figure 9

L = distance from 0 to B along neutral axis of member.

Energy Due to Axial Load

From Eq. 1 letting L = ds and remembering that A and P are variables and letting dU_1 = energy stored in increment of Fig. 9 due to axial load.

$$dU_1 = \frac{P^2 ds}{2\,EA} \tag{1c}$$

$$\text{then } U_1 = \int_0^L \frac{P^2 ds}{2EA} \tag{1d}$$

Energy Due to Shear

From Eq. 2 letting L = ds and remembering that V and A are variables and letting dU_2 = energy stored in increment of Fig. 9 due to shear.

$$dU_2 = \frac{V^2 ds}{2\,GA} \tag{2c}$$

$$\text{then } U_2 = \int_0^L \frac{V^2 ds}{2\,GA} \tag{2d}$$

Energy Due to Bending

From Eq. 3 letting L = ds and remembering that M and I are variables and letting dU_4 = energy stored in increment of Fig. 9 due to bending.

$$dU_3 = \frac{M^2 ds}{2EI} \tag{3c}$$

$$\text{then } U_3 = \int_0^L \frac{M^2 ds}{2EI} \tag{3d}$$

Energy Due to Torsion

From Eq. 4 letting L = ds and remembering that T and J are variables and letting dU_4 = energy stored in increment of Fig. 9 due to torsion.

$$dU_4 = \frac{T^2 ds}{2GJ} \tag{4c}$$

$$\text{then } U_4 = \int_0^L \frac{T^2 ds}{2GJ} \tag{4d}$$

Total Energy Curved Member

Varying Load

Varying Section

Let dU = total energy stored in increment = lb. in.

Then in general

$$dU = dU_1 + dU_2 + dU_3 + dU_4$$

$$= \frac{P^2 ds}{2\,EA} + \frac{V^2 ds}{2\,GA} + \frac{M^2 ds}{2\,EI} + \frac{T^2 ds}{2\,GJ} \tag{5c}$$

then U = total energy stored in member

$$U = \int_0^L \frac{P^2 ds}{2\,EA} + \int_0^L \frac{V^2 ds}{2\,GA} + \int_0^L \frac{M^2 ds}{2\,EI} + \int_0^L \frac{T^2 ds}{2\,GJ} \tag{5d}$$

In the above equations P, V, M, T, A, I and J may be variables.

Castigliano Theorem

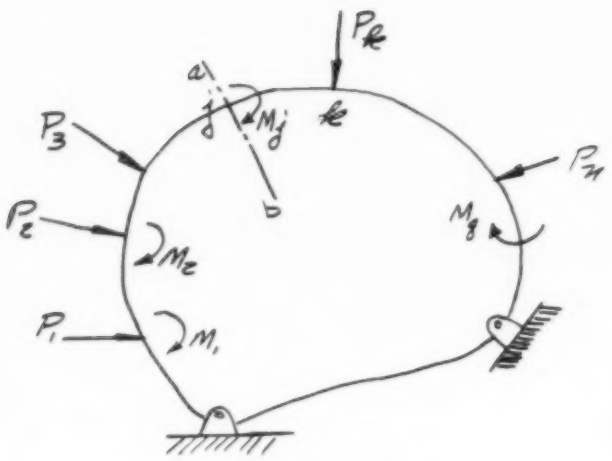

Figure 10

Let U = energy stored in body due to application of loads shown in Fig. 10

then $U = f(P_1, P_2, P_3 - - - P_k - - - P_n - - - M_1, M_2 - - - M_j - - - M_q)$

Let a small increase dP_k be applied to body at point k and in the direction of P_k. See Fig. 10a.

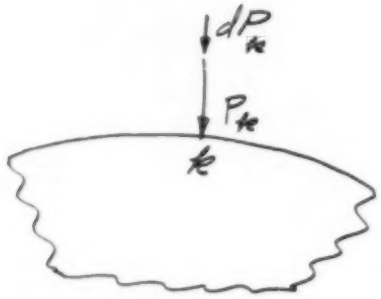

Figure 10a

The strain energy will be increased and the total amount is now designated by U^1:

$$\text{then } U^1 = U + \frac{\partial U}{\partial P_k} \cdot dP_k$$

Suppose now that the small load dP_k is first placed upon the body and that after this the other loads are superimposed.

The total strain energy will be the same as in the first case, as the order in which the loads are applied is of no consequence.

Considering the strain energy stored in the body from this second standpoint, and letting δ_k be the deflection due to the application of the loads $P_1, P_2, P_3, - - - P_k - - - P_n, M_1, M_2, - - - M_j - - - M_q$ the value of U^1 may be written:

$$U^1 = (dP_k)\ \delta_k + U$$

Since the two values of U^1 are equal

$$U + \frac{\partial U}{\partial P_k} dP_k = U + (d P_k)\ \delta_k$$

or

$$\frac{\partial U}{\partial P_k} \cdot dP_k = (d P_k) \cdot S_k$$

or

$$\delta_k = \frac{\partial U}{\partial P_k} \tag{6}$$

Since $U = U_1 + U_2 + U_3 + U_4$

(See Eqs. 5, 5b, and 5d)

an expression for δ_k may be written as follows:

$$\delta_k = \frac{\partial U}{\partial P_k} = \frac{\partial U_1}{\partial P_k} + \frac{\partial U_2}{\partial P_k} + \frac{\partial U_3}{\partial P_k} + \frac{\partial U_4}{\partial P_k} \tag{6a}$$

Again considering Fig. 10 let a small increment of moment $d M_j$ be applied to the body at point j and in the direction of M_j. See Fig. 10b.

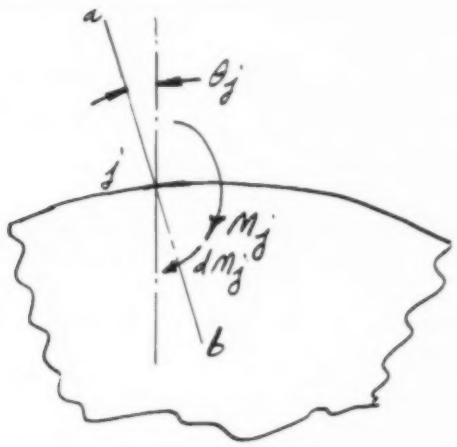

Figure 10b

Now

$$U^1 = U + \frac{\partial U}{\partial M_j} \cdot d M_j$$

assuming that $d M_j$ is applied first and after this, applying the loads of Fig. 10 and letting θ_j be the rotation of any line ab through j due to the application of the system shown in Fig. 10, U^1 may now be written:

$$U^1 = (d M_j)\ \theta_j + U$$

and since the two values of U^1 are equal

$$U + \frac{\partial U}{\partial M_j} \cdot d M_j = U + (d Mj)\ \theta_j$$

or $$\frac{\partial U}{\partial M_j} \cdot dM_j = (d\,M_j)\ \theta_j$$

or $$\theta_j = \frac{\partial U}{\partial M_j} \tag{7}$$

and since $U = U_1 + U_2 + U_3 + U_4$
(See Eqs. 5, 5b and 5d)

an expression for θ_j may be written as follows:

$$\theta_j = \frac{\partial U}{\partial M_j} = \frac{\partial U_1}{\partial M_j} + \frac{\partial U_2}{\partial M_j} + \frac{\partial U_3}{\partial M_j} + \frac{\partial U_4}{\partial M_j} \tag{7a}$$

Example #1

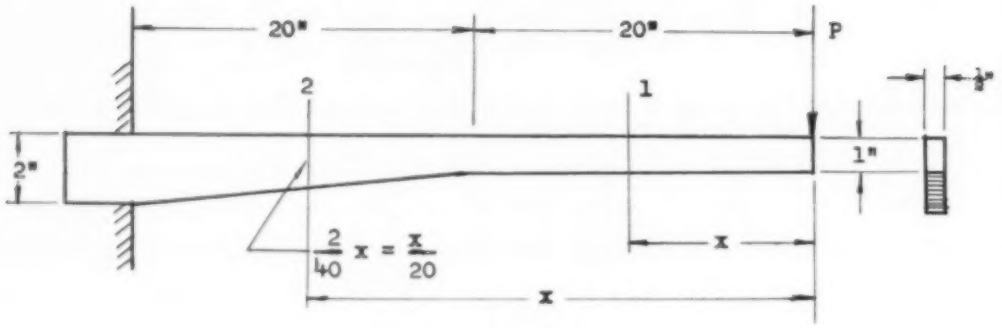

Figure 11

Problem: Determine the deflection at the loaded end (Algebraic Solution)

$$U_1 = \int_0^{20} \frac{M_1^2 dx}{2EI_1} \qquad M_1 = Px \qquad I_1 = \frac{1}{12} \times \frac{1}{2} (1)^3 = \frac{1}{24}$$

$$U_1 = \int_0^{20} \frac{P^2x^2dx}{2\,E\,\frac{1}{24}} = \frac{12P^2}{E} \int_0^{20} x^2dx = \frac{12\,P^2}{E}\left(\frac{x^3}{3}\right)\Bigg]_0^{20}$$

$$= \frac{12\,P^2}{3\,E}(8000) = \frac{32000\,P^2}{E}$$

$$U_2 = \int_{20}^{40} \frac{M_2^2 dx}{2EI_2} \qquad M_2 = Px \qquad I_2 = \frac{1}{12} \times \frac{1}{2} \times \left(\frac{x}{20}\right)^3$$

$$U_2 = \int_{20}^{40} \frac{P^2x^2dx}{2E\,\frac{1}{24}\,\frac{x^3}{8000}} = \frac{96000\,P^2}{E} \int_{20}^{40} \frac{dx}{x}$$

$$U_2 = \frac{96000\ P^2}{E} (\log x)\Big]_{20}^{40} = \frac{96000\ P^2}{E} (\log_e 40 - \log_e 20)$$

$$= \frac{96000\ P^2}{E} \log_e \frac{40}{20} = \frac{96000\ P^2}{E} \log_e 2$$

$$U = U_1 + U_2 = \frac{32000\ P^2}{E} + \frac{96000\ P^2}{E} (\log_e 2)$$

$$= \frac{32000\ P^2}{E} (1 + 3 \log_e 2)$$

$$\delta_P = \frac{\partial U}{\partial P} = \frac{2 \cdot 32000\ P}{E} (1 + \log_e 2^3)$$

$$= \frac{64000\ P}{E} (1 + \log_e 8) = \frac{64000}{30(10)^6} (1 + 2.079)\ P$$

$$= \frac{64 \cdot 3.079}{30{,}000} P$$

Let $P = 100$ #

then $$\delta_P = \frac{64 \cdot 3.079 \cdot 100}{30{,}000} = \underline{\underline{.657"}}$$

$$S_{20} = \frac{100 \cdot 20}{\frac{1}{6} \cdot \frac{1}{2} (1)^2} = 24000\ \#/\text{in.}^2$$

$$S_{40} = \frac{100 \cdot 40}{\frac{1}{6} \cdot \frac{1}{2} \cdot (2)^2} = 12000\ \#/\text{in.}^2$$

Example # 2

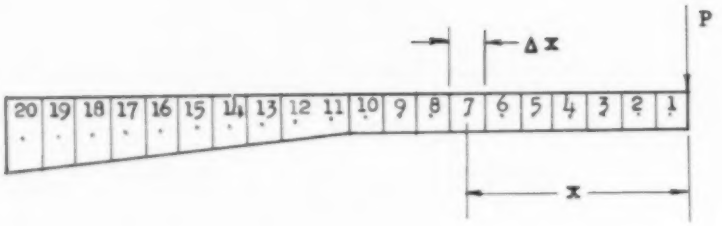

Figure 12

Problem: Solve Example #1 Graphically.

$$\Delta x = \frac{40}{20} = 2" \qquad P = 100\ \#$$

$$U = \int \frac{M^2 dx}{2EI}$$

$$\delta_P = \frac{\partial U}{\partial P} = \int \frac{M}{EI}\frac{\partial M}{\partial P}\, dx = \sum_0^{40} \frac{M}{EI}\frac{\partial M}{\partial P} \cdot \Delta x$$

$$M = Px \qquad \frac{\partial M}{\partial P} = x$$

$$\delta_P = \sum_0^{40} \frac{Px}{EI} x \cdot \Delta x = \sum_0^{40} \frac{Px^2}{EI} \Delta x = \frac{P}{E} \Delta x \sum_0^{40} \frac{x^2}{I}$$

$$= \frac{100}{30(10)^6} \cdot 2 \sum_0^{40} \frac{x^2}{I} = \frac{2}{30(10)^4} \sum_0^{40} \frac{x^2}{I}$$

From Table 1 Pg. 37 $\sum_0^{40} \frac{x^2}{I} = 98433$

Then $\delta_P = \frac{2}{30{,}000} \cdot 98433 = \underline{\underline{.65622}}$

from algebraic solution Pg. 34 $\delta_p = .657"$

% error $= \frac{.657 - .656}{.657} \cdot 100 = .15$

Table 1

No.	X	X^2	I	X^2/I
1	1	1	1/24	24
2	3	9	1/24	216
3	5	25	1/24	600
4	7	49	1/24	1176
5	9	81	1/24	1944
6	11	121	1/24	2904
7	13	169	1/24	4056
8	15	225	1/24	5400
9	17	289	1/24	6936
10	19	361	1/24	8664
11	21	441	.0482	9143
12	23	529	.0634	8348
13	25	625	.0814	7680
14	27	729	.1025	7111
15	29	841	.1270	6621
16	31	961	.1552	6194
17	33	1089	.1872	5818
18	35	1225	.2233	5486
19	37	1369	.2638	5189
20	39	1521	.3090	4923
		$\sum_{0}^{40} X^2/I$	=	98433

$$I_{1-10} = \frac{1}{24}$$

$$\frac{x^2}{I_{1-10}} = 24x^2$$

$$I_{11-20} = \frac{x^3}{192000}$$

$$\frac{x^2}{I_{11-20}} = \frac{192000}{x}$$

Example #3

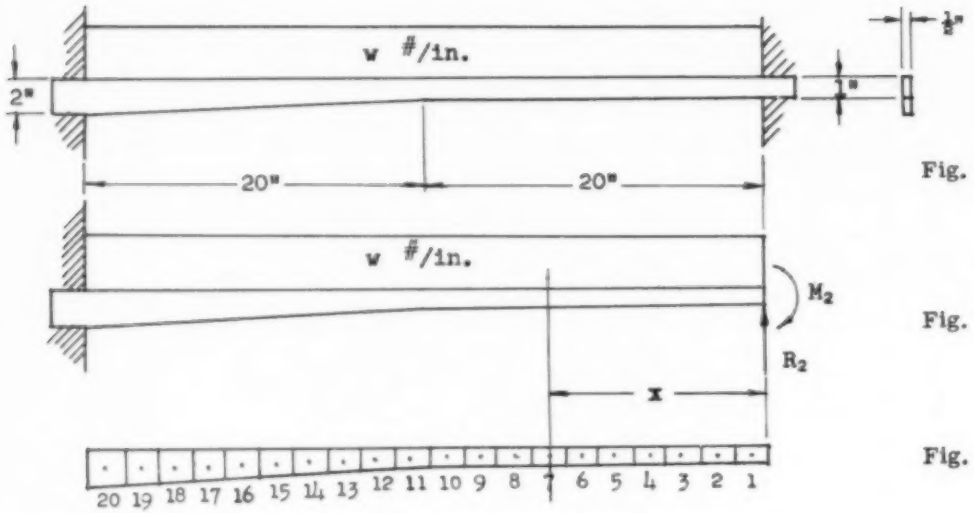

Fig. 13a

Fig. 13b

Fig. 13c

Problem: Determine the reactions of the beam shown. Also determine the max. moment and max. stress.

$$M = R_2x - M_2 - \frac{wx^2}{2} \quad , \quad \frac{\partial M}{\partial R_2} = x \quad , \quad \frac{\partial M}{\partial M_2} = -1$$

$$U = \int_0^{40} \frac{M^2 dx}{2EI} \quad \text{(From Eq. 3b)}$$

$$\theta_2 = \frac{\partial U}{\partial M_2} = \int_0^{40} \frac{M}{EI} \frac{\partial M}{\partial M_2} dx = 0 \qquad (A)$$

$$\delta_2 = \frac{\partial U}{\partial R_2} = \int_0^{40} \frac{M}{EI} \frac{\partial M}{\partial R_2} dx = 0 \qquad (B)$$

for graphical integration

$$\sum_0^{20} \frac{(R_2x - M_2 - \frac{wx^2}{2})(-1)\,\Delta x}{EI} = 0 \quad \text{or} \quad \sum_0^{20} \left(\frac{R_2x}{I} - \frac{M_2}{I} - \frac{wx^2}{2I} \right) = 0 \qquad (A_1)$$

$$\sum_0^{20} \frac{(R_2x - M_2 - \frac{wx^2}{2})(x)\Delta x}{EI} = 0 \quad \text{or} \quad \sum_0^{20} \left(\frac{R_2x^2}{I} - \frac{M_2x}{I} - \frac{wx^3}{2I} \right) = 0 \qquad (B_1)$$

then

$$R_2 \sum_{0}^{20} \frac{x}{I} - M_2 \sum_{0}^{20} \frac{1}{I} - \frac{w}{2} \sum_{0}^{20} \frac{x^2}{I} = 0 \qquad (A_2)$$

$$R_2 \sum_{0}^{20} \frac{x^2}{I} - M_2 \sum_{0}^{20} \frac{x}{I} - \frac{w}{2} \sum_{0}^{20} \frac{x^3}{I} = 0 \qquad (B_2)$$

Using coefficients from Table 3.

$$4796.5\ R_2 - 329.72\ M_2 = \frac{w}{2} \times 98432.2 = 0 \qquad (A_3)$$

$$98432.2\ R_2 - 4796.5\ M_2 = \frac{w}{2} \times 2{,}397{,}600 = 0 \qquad (B_3)$$

$$R_2 - .06874\ M_2 = 10.2608\ w \qquad (A_4)$$

$$R_2 - .04873\ M_2 = 12.1789\ w \qquad (B_4)$$

$$-.02001\ M_2 = -1.9181\ w \qquad (C)$$

$$M_2 = \underline{95.857\ w}$$

$$14.5472\ R_2 - M_2 = 149.266\ w$$

$$20.5217\ R_2 - M_2 = 249.932\ w$$

$$5.9745\ R_2 = 100.666\ w$$

$$R_2 = \underline{16.849\ w}$$

At any point a distance x from the right end

$$M = -95.86\ w + 16.85\ w\ x - \frac{wx^2}{2}$$

$$= (-95.86 + 16.85\ x - \frac{x^2}{2})\ w \qquad (D)$$

Table 3 shows the coordinates of Eq. D and Fig. 14 the corresponding moment diagram.

Table 2

No.	X	X^2	X^3	I	1/I	X/I	X^2/I	X^3/I
1	1	1	1	1/24	24	24	24	24
2	3	9	27	1/24	24	72	216	648
3	5	25	125	1/24	24	120	600	3000
4	7	49	343	1/24	24	168	1176	8232
5	9	81	729	1/24	24	216	1944	17496
6	11	121	1331	1/24	24	264	2904	31944
7	13	169	2197	1/24	24	312	4056	52728
8	15	225	3375	1/24	24	360	5400	81000
9	17	289	4913	1/24	24	408	6936	117912
10	19	361	6859	1/24	24	456	8664	164616
11	21	441	9261	.0482	20.732	435.374	9142.9	192000
12	23	529	12167	.0634	15.780	362.949	8347.8	192000
13	25	625	15625	.0814	12.288	307.200	7680.0	192000
14	27	729	19683	.1025	9.755	263.374	7111.1	192000
15	29	841	24389	.1270	7.872	228.300	6620.7	192000
16	31	961	29791	.1552	6.445	199.792	6193.5	192000
17	33	1089	35937	.1872	5.343	176.309	5818.2	192000
18	35	1225	42875	.2233	4.478	156.735	5485.7	192000
19	37	1369	50653	.2638	3.790	140.248	5189.2	192000
20	39	1521	59319	.3090	3.237	126.233	4923.1	192000
				Σ =	329.720	4796.514	98432.2	2397600

Table 3

X	M
0	- 95.86 w
5	- 24.11 w
10	+ 22.64 w
15	+ 44.39 w
20	+ 41.14 w
25	+ 12.89 w
30	- 40.36 w
35	- 118.61 w
40	- 221.86 w

Fig 14 shows the moment at various points on the beam. From this figure it appears that the max. stress occurs at either the left end or the right end of the beam.

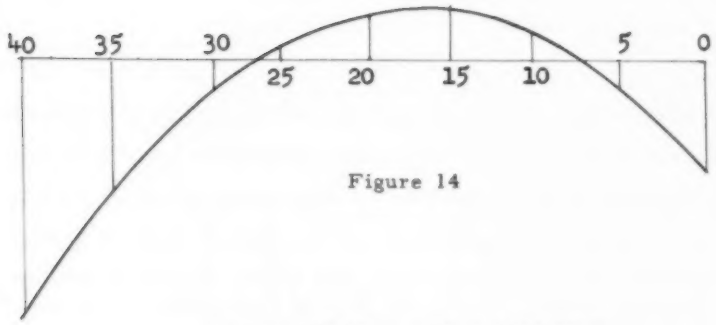

Figure 14

$S_0 = 95.86\ w \times 12 = 1150.32\ w$

$S_{40} = 221.86\ w \times 3 = 665.58\ w$

Then $S_0 = S_{max.}$ and if $w = 20$ #/in

$S_{max.} = 1150.32 \times 20 = 23000$ #/in^2

THE EFFECT OF SLICING ON RESIDUAL STRESSES IN PLATES

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Introduction

In determining the residual stresses in a plate, the usual procedure is to remove a thin strip from the plate and then to determine the residual stresses in the strip. However, the strip stresses will be different from the original plate stresses, and in order to convert strip stresses into plate stresses it becomes necessary to solve a boundary value problem involving the biharmonic partial differential equation. A discussion of this boundary value problem, together with a method of arriving at some satisfactory solutions, is herewith presented.

Mathematical Statement of the Problem

Let P be a rectangular plate of thickness t, as shown in Figure 1 below:

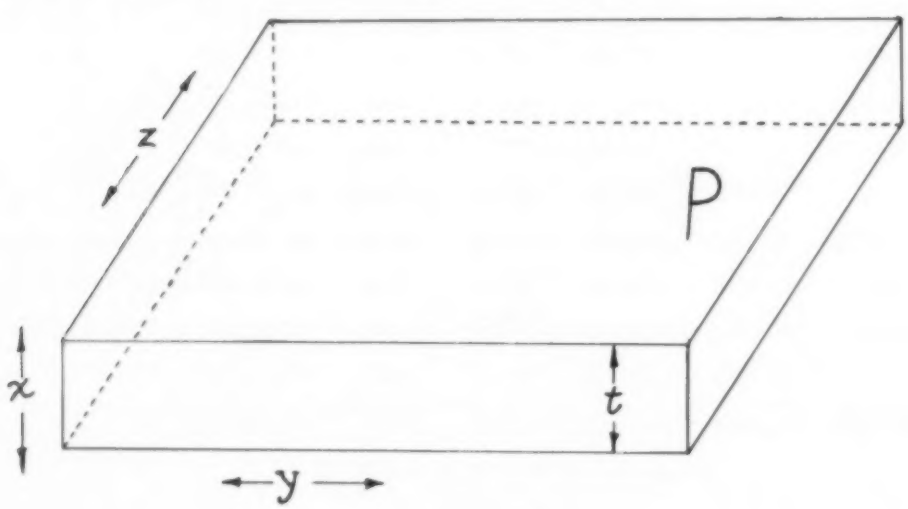


Figure 1

Let us choose a coordinate system in which the x-axis extends in the up and down direction, the y-axis to the left and right, and the z-axis forward and backward. Next, suppose a slice of width a is removed from the middle of the plate in such a manner that the length of the slice extends in the z direction, as shown in Figure 2.

This slicing will relieve a direct stress $\sigma_y\,(x, \frac{a}{2}, z)$ on the faces $V_1V_2V_3V_4$ and $V_5V_6V_7V_8$ of the strip. The shear stresses on $V_1V_2V_3V_4$ and $V_5V_6V_7V_8$ also vanish. Furthermore, the top and bottom faces are free from normal and shear stresses. As a result of this slicing operation the residual stress σ_z of the strip will be different from the residual stress σ_z of the plate. In order to determine the difference between these σ_z stresses, we shall start with a stress-free strip and apply a normal stress $\sigma_y\,(x, \frac{a}{2}, z)$ to each of the two faces $V_1V_2V_3V_4$ and $V_5V_6V_7$

V_8, and then proceed to determine the stress components σ_x, σ_y, and σ_z induced in the strip according to the theory of elasticity. In particular, the component σ_z so determined will represent the difference between the σ_z residual stresses in the plate and strip, which is what we are seeking.

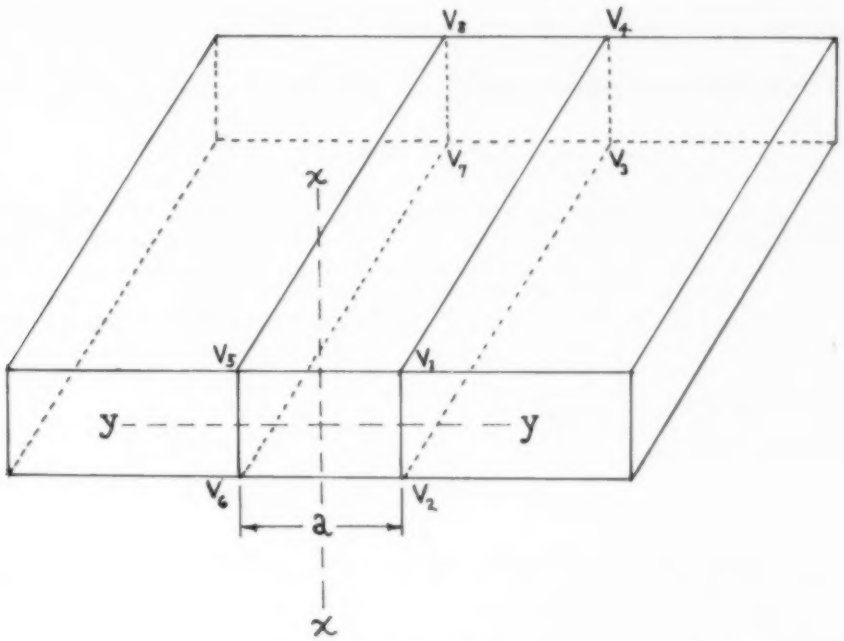


Figure 2

σ_y must be symmetrical with respect to the x-axis, since we are assuming that the y-stress in the original plate does not vary with y. The stress $\sigma_y\,(x, \frac{a}{2}, z)$ as applied to the faces V_1V_2 V_3V_4 and $V_5V_6V_7V_8$ must be balanced both for load and moment, since it is a residual stress which exists without external loading on the original plate. Hence, if we make $\sigma_y\,(x, \frac{a}{2}, z)$ symmetrical with respect to the y-axis, i.e., an even function of x, and balance it for load, the moments will also be in balance and we will still have complete freedom to specify the behavior of $\sigma_y\,(x, \frac{a}{2}, z)$ near the top surface.

Since the problem is one of plane strain, we shall confine our attention to the rectangle $V_1V_2V_6V_5$, with the origin taken at its center. The boundary conditions of the problem are the following:

Along V_1V_5 and V_2V_6 : $\sigma_x\,(\pm\frac{t}{2}, y) = 0$ (1)

Along V_1V_2 and V_5V_6 : $\sigma_y\,(x, \pm\frac{a}{2}) = h(x)$, where $h(x)$ is any specified even function of x, such that $\int_0^{t/2} h(x)dx = 0$. (2)

Along all four borders : $\tau_{xy} = 0$, i.e., $\tau_{xy}\,(\pm\frac{t}{2}, y) = 0$, and $\tau_{xy}\,(x, \pm\frac{a}{2}) = 0$ (3)

Our problem, therefore, reduces to that of finding an Airy's stress function, ϕ, satisfying the partial differential equation

$$\frac{\partial^4 \phi}{\partial x^4} + 2\frac{\partial^4 \phi}{\partial x^2 \partial y^2} + \frac{\partial^4 \phi}{\partial y^4} = 0,$$

subject to the above mentioned boundary conditions.

In terms of ϕ the boundary conditons become

$$\frac{\partial^2 \phi}{\partial y^2} = 0 \quad \text{at} \quad x = \pm \frac{t}{2}, \quad \text{since} \quad \sigma_x = \frac{\partial^2 \phi}{\partial y^2}$$

$$\frac{\partial^2 \phi}{\partial x^2} = h(x) \quad \text{at} \quad y = \pm \frac{a}{2}, \quad \text{since} \quad \sigma_y = \frac{\partial^2 \phi}{\partial x^2}$$

$$\frac{\partial^2 \phi}{\partial x \partial y} = 0 \quad \text{at} \quad x = \pm \frac{t}{2}, \quad \text{since} \quad \tau_{xy} = -\frac{\partial^2 \phi}{\partial x \partial y}$$

$$\frac{\partial^2 \phi}{\partial x \partial y} = 0 \quad \text{at} \quad y = \pm \frac{a}{2}.$$

Some Particular Solutions

We arrive at some particular solutions to the problem by assuming a stress function of the form

$$\phi(x,y) = F(x,y) + \sum_{n=1}^{\infty} \left[A_n \cosh \frac{2n\pi x}{a} + B_n \left(\frac{2n\pi x}{a}\right) \sinh \frac{2n\pi x}{a} \right] \cos \frac{2n\pi y}{a},$$

where F(x,y) may be any function such that

$$\frac{\partial^4 F}{\partial x^4} + 2\frac{\partial^4 F}{\partial x^2 \partial y^2} + \frac{\partial^4 F}{\partial y^4} = 0,$$

and such that

$$\frac{\partial^2 F}{\partial x \partial y} = 0 \quad \text{at} \quad y = \pm \frac{a}{2}.$$

The values of the coefficients A_n and B_n will be determined by our particular choice of F(x,y). It is desirable to make F(x,y) an even function of both x and y. This will make $\frac{\partial^2 F}{\partial x \partial y}$ an odd function of both x and y.

The second partial derivatives of ϕ (x,y) are

$$\frac{\partial^2\phi}{\partial x^2} = \frac{\partial^2 F}{\partial x^2} + \sum_{n=1}^{\infty}\left[\left(\frac{2n\pi}{a}\right)^2(A_n+2B_n)\cosh\frac{2n\pi x}{a} + \left(\frac{2n\pi}{a}\right)^2\cdot B_n\cdot\left(\frac{2n\pi x}{a}\right)\sinh\frac{2n\pi x}{a}\right]\cos\frac{2n\pi y}{a} \quad (4)$$

$$\frac{\partial^2\phi}{\partial y^2} = \frac{\partial^2 F}{\partial y^2} - \sum_{n=1}^{\infty}\left[\left(\frac{2n\pi}{a}\right)^2 A_n\cosh\frac{2n\pi x}{a} + \left(\frac{2n\pi}{a}\right)^2\cdot B_n\cdot\left(\frac{2n\pi x}{a}\right)\sinh\frac{2n\pi x}{a}\right]\cos\frac{2n\pi y}{a} \quad (5)$$

$$\frac{\partial^2\phi}{\partial x\,\partial y} = \frac{\partial^2 F}{\partial x\partial y} - \sum_{n=1}^{\infty}\left[\left(\frac{2n\pi}{a}\right)^2(A_n+B_n)\sinh\frac{2n\pi x}{a} + \left(\frac{2n\pi}{a}\right)^2\cdot B_n\cdot\left(\frac{2n\pi x}{a}\right)\cosh\frac{2n\pi x}{a}\right]\sin\frac{2n\pi y}{a} \quad (6)$$

The boundary condition $\sigma_x(\pm\frac{t}{2}, y) = 0$ implies that

$$\left[\frac{\partial^2 F}{\partial y^2}\right]_{x=\pm\frac{t}{2}} = \sum_{n=1}^{\infty}\left[\left(\frac{2n\pi}{a}\right)^2\cdot A_n\cdot\cosh\frac{n\pi t}{a} + \left(\frac{2n\pi}{a}\right)^2\cdot B_n\cdot\left(\frac{n\pi t}{a}\right)\sinh\frac{n\pi t}{a}\right]\cos\frac{2n\pi y}{a} \quad (7)$$

The boundary condition $\tau_{xy}(\pm\frac{t}{2}, y) = 0$ implies that

$$\left[\frac{\partial^2 F}{\partial x\partial y}\right]_{x=\pm\frac{t}{2}} = \sum_{n=1}^{\infty}\left[\left(\frac{2n\pi}{a}\right)^2(A_n+B_n)\sinh\frac{n\pi t}{a} + \left(\frac{2n\pi}{a}\right)^2\cdot B_n\cdot\left(\frac{n\pi t}{a}\right)\cosh\frac{n\pi t}{a}\right]\sin\frac{2n\pi y}{a} \quad (8)$$

The boundary condition $\tau_{xy}(x, \pm\frac{a}{2}) = 0$ is already taken care of, since both parts of (6) vanish at $y = \pm\frac{a}{2}$. Equations (7) and (8) tell us that if we choose A_n and B_n so as to make the right hand members the Fourier expansions of

$$\left[\frac{\partial^2 F}{\partial y^2}\right]_{x=\pm\frac{t}{2}} \quad \text{and} \quad \left[\frac{\partial^2 F}{\partial x\,\partial y}\right]_{x=\pm\frac{t}{2}},$$

respectively, then the boundary conditions $\sigma_x(\pm\frac{t}{2}, y) = 0$ and $\tau_{xy}(\pm\frac{t}{2}, y) = 0$ will be satisfied. The determination of A_n and B_n will therefore reduce to the solving of a pair of linear algebraic equations. After finding the values of A_n and B_n, we can substitute them into (4), and examine

$$\left[\frac{\partial^2\phi}{\partial x^2}\right]_{y=\pm\frac{a}{2}} = \sigma_y(x, \pm\tfrac{a}{2})$$

to see whether or not it is a satisfactory loading function h(x).

The Change in Z-Stress due to Slicing

According to Hooke's Law for an isotripic medium, the strain component in the z-direction is given by

$$\epsilon_z = \frac{1}{E}\left[\sigma_z - \nu(\sigma_x + \sigma_y)\right]$$

where ν is Poisson's ratio. In plane strain this strain component is zero. Hence,

$$\frac{1}{E}\left[\sigma_z - \nu(\sigma_x + \sigma_y)\right] = 0$$

$$\text{or,} \quad \sigma_z = \nu(\sigma_x + \sigma_y).$$

This latter expression represents the amount by which σ_z in the plate exceeds σ_z in the strip at any point (x,y). Let us, therefore, replace σ_z by $\Delta\sigma_z$ and write

$$\Delta\sigma_z = \nu(\sigma_x + \sigma_y).$$

At any one x-level, the average amount by which σ_z of the plate exceeds σ_z of the strip will then be given by the integral

$$\overline{\Delta\sigma_z} = \frac{1}{a}\int_{-\frac{a}{2}}^{\frac{a}{2}} \Delta\sigma_z \, dy$$

This becomes

$$\frac{1}{a}\int_{-\frac{a}{2}}^{\frac{a}{2}} \nu(\sigma_x + \sigma_y)\, dy = \frac{\nu}{a}\int_{-\frac{a}{2}}^{\frac{a}{2}} \sigma_x \, dy + \frac{\nu}{a}\int_{-\frac{a}{2}}^{\frac{a}{2}} \sigma_y \, dy$$

But

$$\int_{-\frac{a}{2}}^{\frac{a}{2}} \sigma_x \, dy = 0,$$

since σ_x is a balanced loading function.

Hence,

$$\overline{\Delta\sigma_z} = \frac{1}{a}\int_{-\frac{a}{2}}^{\frac{a}{2}} \Delta\sigma_z \, dy = \frac{\nu}{a}\int_{-\frac{a}{2}}^{\frac{a}{2}} \sigma_y \, dy = R\left(x, \frac{a}{t}\right)\cdot\nu\,\sigma_y\left(x, \frac{a}{2}\right),$$

where $R(x,\frac{a}{t})$ is a factor somewhere between 0 and 1. In particular,

If $R(x, \frac{a}{t}) = 1$, we say that there is 100% relaxation.

If $R(x, \frac{a}{t}) = .5$, we say that there is 50% relaxation.

If $R(x, \frac{a}{t}) = 0$, we say that there is 0% relaxation.

etc. etc. etc.

The ratio

$$\frac{\overline{\Delta\sigma_z}}{\nu\,\sigma_y\left(x, \frac{a}{2}\right)} = R\left(x, \frac{a}{t}\right)$$

shall hereafter be referred to as the relaxation ratio.

Illustrations of the Method

Case I: The simplest function F(x,y) that we can choose is obviously that one for which $\frac{\partial^2 F}{\partial x \partial y} = 0$ (Identically). This leads to an F of the form

$$F(x,y) = (-x^4 + c_3 x^2) + (y^4 + k_3 y^2),$$

where c_3 and k_3 are constants to be determined from the boundary conditions. From this special function F , we obtain

$$\sigma_y(x, \pm \tfrac{a}{2}) = h(x) = (-12x^2 + t^2) + \sum_{n=1}^{\infty} (-1)^n \left[\left(\frac{2n\pi}{a}\right)^2 (A_n + 2B_n) \cosh \frac{2n\pi x}{a} + \left(\frac{2n\pi}{a}\right)^2 \cdot B_n \cdot \left(\frac{2n\pi x}{a}\right) \sinh \frac{2n\pi x}{a} \right].$$

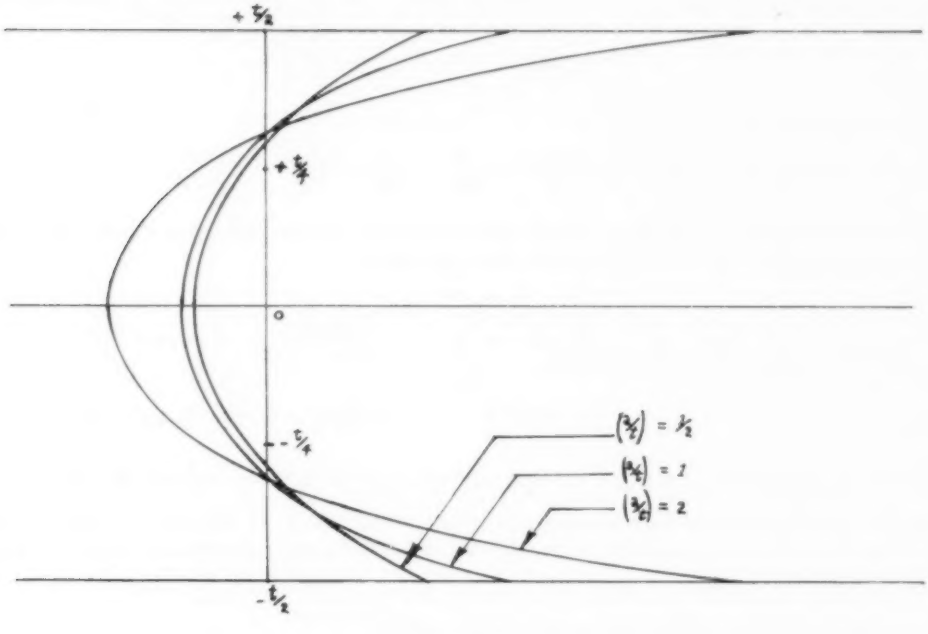


Figure 3

This loading function is shown in Figure 3 for various values of $(\frac{a}{t})$. We see that it typifies the simplest type of residual stress distribution. The general formula for σ_y (x,y) is found to be

$$\sigma_y(x,y) = (-12x^2 + t^2) + \sum_{n=1}^{\infty} \left[\left(\frac{2n\pi}{a}\right)^2 (A_n + 2B_n) \cosh \frac{2n\pi x}{a} + \left(\frac{2n\pi}{a}\right)^2 \cdot B_n \cdot \left(\frac{2n\pi x}{a}\right) \sinh \frac{2n\pi x}{a} \right] \cos \frac{2n\pi y}{a}$$

In order to compute the relaxation ratio, we must determine the average y-stress from $y = -\frac{a}{2}$ to $y = \frac{a}{2}$ for each x-level. This average y-stress is equal to

$$\bar{\sigma}(x) = \frac{1}{a}\int_{-\frac{a}{2}}^{\frac{a}{2}} \sigma_y(x,y)\, dy = \frac{1}{a}\int_{-\frac{a}{2}}^{\frac{a}{2}} (-12x^2 + t^2)dy = t^2 - 12x^2 .$$

It should be noted that since the trigonometric series portion of the expression for the y-stress is of such a nature that its integral vanishes at the two limits, it need not be included in the above integration. We are primarily interested in the relaxation ratio at the top surface, i.e. at $x = \frac{t}{2}$. This will be given by the formula

$$R\left(\tfrac{t}{2}, \tfrac{a}{t}\right) = \frac{\bar{\sigma}\left(\frac{t}{2}\right)}{\sigma_y\left(\frac{t}{2}, \frac{a}{2}\right)} = \frac{-2t^2}{\sigma_y\left(\frac{t}{2}, \frac{a}{2}\right)} .$$

In curve (I) of Figure 5 this particular relaxation ratio is plotted versus $\frac{a}{t}$.

Case II: As a second example, consider the case in which the function F is of such a nature that

$$\frac{\partial^2 F}{\partial x\, \partial y} = x\left(y^3 - \frac{a^2 y}{4}\right) .$$

This implies that

$$F(x,y) = \frac{x^2y^4}{8} - \frac{a^2x^2y^2}{16} - \frac{x^6}{120} - \frac{y^6}{60} + \frac{a^2y^4}{48} + \frac{C_3 x^2}{2} + \frac{K_3 y^2}{2} ,$$

where the constants C_3 and K_3 are to be determined from the boundary conditions. The general expression for the y-stress in this case becomes

$$\sigma_y(x,y) = \frac{y^4}{4} - \frac{a^2y^2}{8} - \frac{x^4}{4} + \frac{7a^4 + 3t^4}{960} + \sum_{n=1}^{\infty} \left[\left(\frac{2n\pi}{a}\right)^2 (A_n + 2B_n) \cosh \frac{2n\pi x}{a} + \left(\frac{2n\pi}{a}\right)^2 \cdot B_n \cdot \left(\frac{2n\pi x}{a}\right) \sinh \frac{2n\pi x}{a}\right] \cos \frac{2n\pi y}{a}$$

When this is evaluated at $y = \pm\frac{a}{2}$, it becomes the type of loading function shown in Figure 4. It can be seen that there is a steeper stress gradient close to the top surface in this case. This therefore may more closely simulate the residual stress conditions near the surface of certain specimens.

The average y-stress across the width of the strip is

$$\bar{\sigma}(x) = \frac{1}{a}\int_{-\frac{a}{2}}^{\frac{a}{2}} \left(\frac{y^4}{4} - \frac{a^2y^2}{8} - \frac{x^4}{4} + \frac{7a^4 + 3t^4}{960}\right) dy = \frac{t^4 - 80x^4}{320} .$$

Consequently, the relaxation at $x = \frac{t}{2}$ is

$$R\left(\tfrac{t}{2}, \tfrac{a}{t}\right) = \frac{-\frac{t^4}{80}}{\sigma_y\left(\frac{t}{2}, \frac{a}{2}\right)} .$$

Curve (II) of Figure 5 is a plot of this relaxation ratio versus $\frac{a}{t}$.

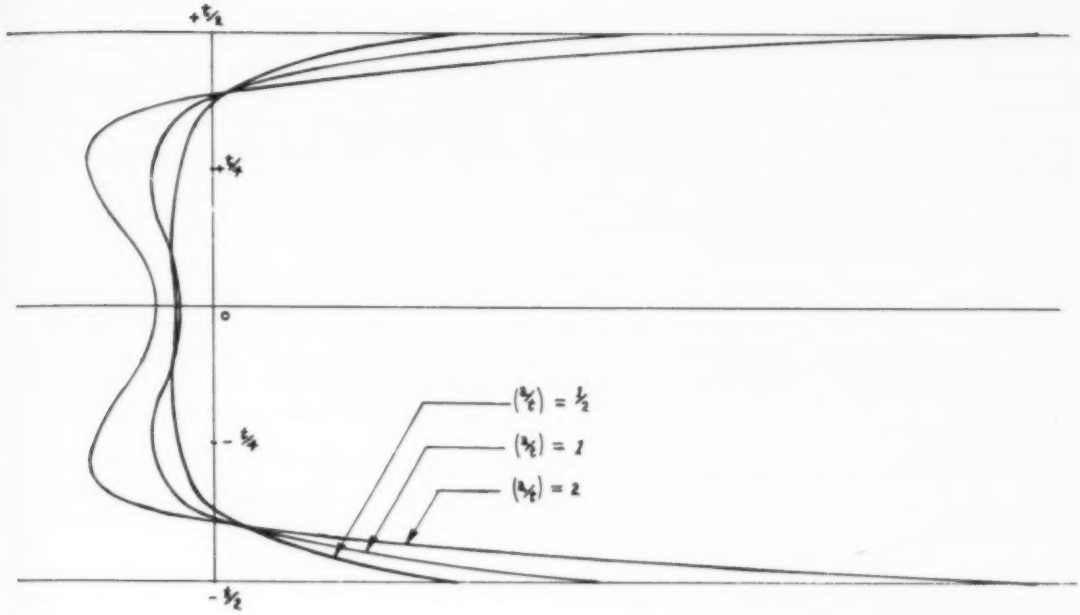


Figure 4

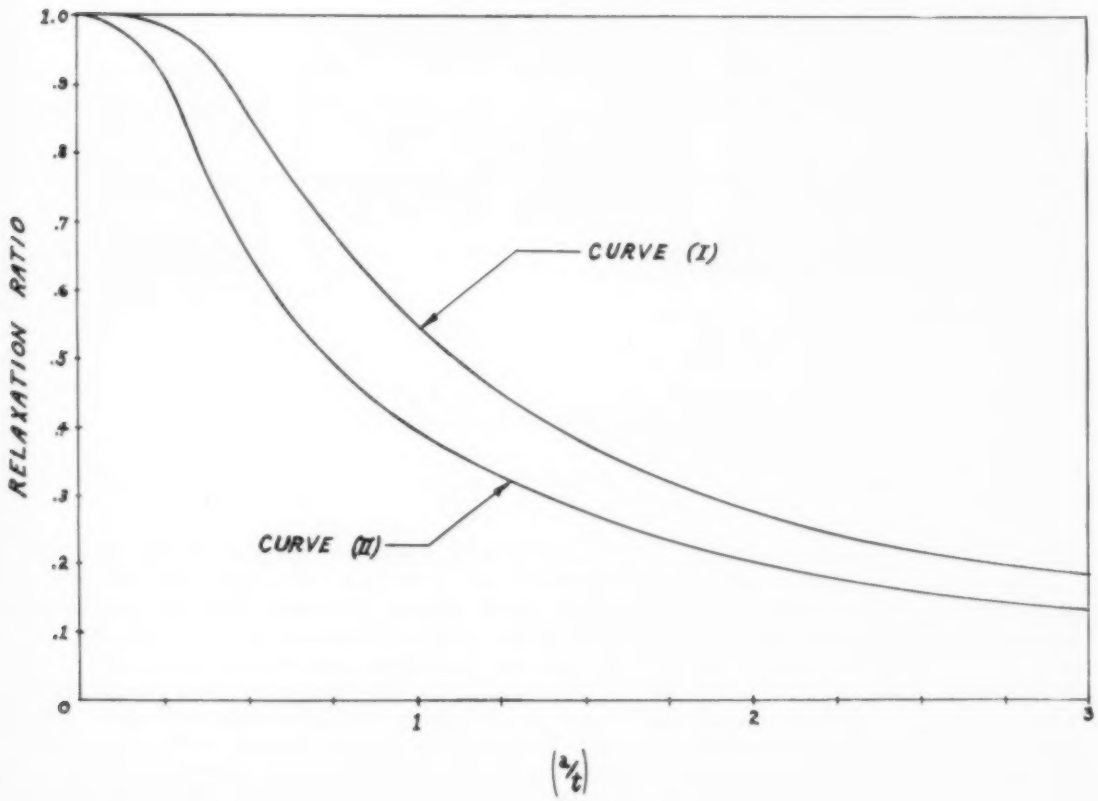


Figure 5

Conclusion

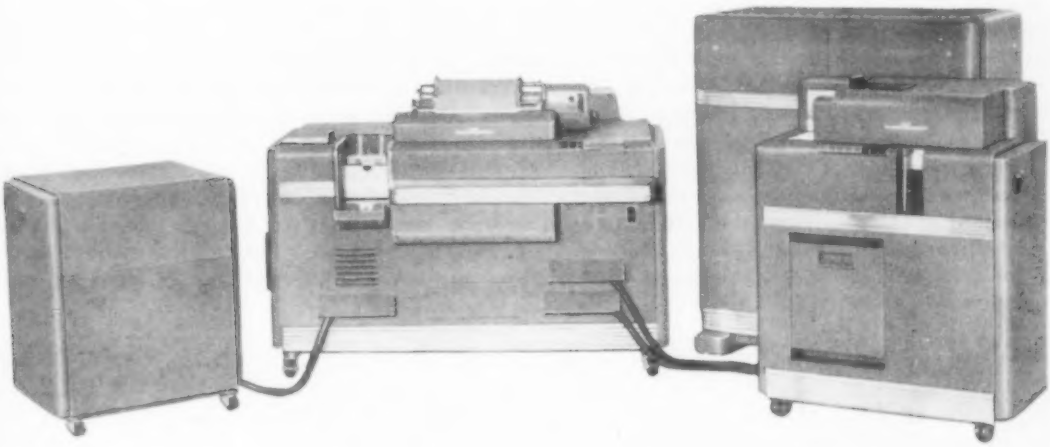
By examining Figure 5 it can be seen that as long as the width of the strip is not more than one-fourth of the thickness of the plate from which it is cut, the amount of relaxation is 90% or greater. In other words, the practicing stress engineer is not committing any gross error when he assumes that the change in Z-stress due to slicing is equal to the released y-stress multiplied by Poisson's ratio, provided that the width of the strip does not exceed one-fourth the thickness of the plate.

THE IBM CARD-PROGRAMMED ELECTRONIC CALCULATOR

JUSTUS CHANCELLOR, III
International Business Machines Corporation

The IBM Card-Programmed Electronic Calculator is a general purpose computer built specifically to solve engineering design and scientific research problems. It is a commercially available model of two preceding larger counterparts the IBM Automatic Sequence Controlled Calculator and the IBM Selective Sequence Electronic Calculator. The Card-Programmed Calculator is referred to in brief as the CPC.

Physically, the CPC consists of four units. These units are all interconnected by electric cables and they function as a single integrated calculator.



IBM Card-Programmed Electronic Calculator

605

941 412 527

The type 412 Accounting Machine is the central unit of the combination. It receives input values and instructions for operation, exercises control over the other units, and records intermediate and final results in printed form. This unit contains counters which can be used to store values or to accumulate totals algebraically. There are eighty of these counters which can be grouped to accommodate 6 ten digit numbers and 1 seven digit number with full algebraic sign control; or, 8 eight digit numbers; or any other desired combination. Both input values and numerical instructions are introduced into the type 412 by means of standard IBM cards. When desired, the machine makes a complete printed record of all of the instructions it has carried out as well as each of the numbers involved in the calculations and the computed results. Printing is accomplished a complete line at a time with 88 type bars working in parallel at a speed of 100 lines per minute. Thus the Accounting Machine has a print-

ing capacity of 8,800 characters per minute. The type bars are divided so that the left 43 are capable of printing any letter of the alphabet as well as all numerals. The remaining 45 type bars print numeric characters only. If the type 418 Accounting Machine, which has 88 type bars each of which contains numerals only, is used instead of the type 412, the printing speed is fifty per cent faster.

The principal function of the type 605 Electronic Calculating Unit is to carry out calculations on numbers transmitted to it from the counters of the 412 as well as from other storage locations. As a digital calculator, the CPC uses an approach to the solution of a given problem which is similar in certain ways to the method in which the particular problem would be attacked if it were solved by hand. Thus the problem is broken down into individual arithmetic steps of the form "A" Operation "B" equals "C". For example, 3 plus 2 equals 5; or, 9 multiplied by 6 equals 54, or, 72 divided by 8 equals 9. These, of course, are simplified cases. Actually, instead of working with one or two digit numbers as illustrated, the CPC works with up to ten digits in each of the values involved in any one arithmetic step. While the largest number of digits in each of the factors which may be conveniently handled in the CPC is ten, there are several different ways in which the calculator may be set-up as a general purpose computer.

One typical set-up of the CPC involves numbers each of which contains eight digits. Using factors of this size it is possible to have available in the type 605 Electronic Calculating Unit several different operations at one time. First of all the machine can add, subtract, multiply, or divide two such eight digit factors and obtain an eight digit result. As well as these simple arithmetic operations, the type 605 is also capable of calculating at electronic speed various special functions which are normally associated with mathematical tables. This includes, square root, sine, cosine, arc sine, arc cosine, hyperbolic sine, hyperbolic cosine, arc tangent, arc hyperbolic tangent, logarithms to the base "e" or base 10, "e" to a given exponent, as well as other special functions. Thus wherever there is a known convergent series which describes a particular function, this function can be included as a single operation in the CPC. Calculating this function at electronic speed is no more difficult for the CPC than is simple multiplication or division. As a result, many calculations normally associated with elaborate table look-up operations in manual computations become simple machine calculations when carried out on the CPC.

Nearly all of the calculations completed on the CPC are accomplished in the type 605 Electronic Calculating Unit. As well as performing all of the calculating operations mentioned above, the type 605 also is capable of completing logical distinctions and can alter its sequence of calculations based on these distinctions. Thus the 605 can change its pattern of calculations in a prescribed way based on the recognition of a positive or negative condition in a given factor or result. It can also distinguish a zero from a non-zero factor or result and can take certain desired action based on this distinction.

When the type 605 has completed a calculation the result is transmitted from the electronic counter to storage locations in the 412 or is printed.

The type 941 Auxiliary Storage Unit is used to supplement the counter capacity of the 412 to store initial values and the results of intermediate calculations. One type 941 unit has the capacity to store 16 ten digit numbers. Additional 941 units are available if additional storage capacity is necessary. Each of the 16 storage registers in one 941 unit is different from a counter in that the storage register is not designed to accumulate the algebraic sum of two successive values added into it. A storage register can contain only one value at one time. If a new value is read into a given storage register any previous value standing in the register is automatically eliminated before the new value is entered.

The type 527 Gang Summary Punch is used for recording intermediate and final results in punched form in standard IBM cards. This form of output of results, of course, supplements

the printed record made by the 412 during the calculations. The punched card form of recording results is useful where answers are to be used in subsequent calculations on IBM equipment or in later operations on the CPC.

Thus the four components, each with its own special functions, are integrated into a single general purpose electronic calculator to form the CPC. The components of the CPC are modified forms of time-proven accounting machines. Therefore the units may be separated and used individually as standard accounting machines. The type 605 and the type 527 may be connected to form an Electronic Calculating Punch. The type 412 can be used separately as an Accounting Machine or it may be used in combination with the type 941 and/or the 527 as a Summary Punch.

It is a simple matter for a design engineer, research scientist, or mathematician to plan his own problems for solution on the CPC. A Planning Chart has been designed to facilitate the coding of a problem for solution on the CPC.

CARD-PROGRAMMED ELECTRONIC CALCULATOR PLANNING CHART

APPLICATION

| OPERATION | PUNCH INTO INSTRUCTION CARDS | | |

The group of columns headed "Counter Groups" is used for recording those values to be stored or accumulated algebraically in particular counter groups. Thus the use of the counters in the 412 is planned so that the proper values are retained in the correct locations.

The final group of columns is labeled "Auxiliary Storage Registers". There are 16 of these columns, one for each storage register. Each register will retain a ten digit number. When a particular storage register is used to contain a given factor or result in a calculation the mathematical symbol representing this number is written in the proper column.

Thus by using the right hand half of the planning chart the individual coding the problem keeps track of which values and results are stored in what storage locations and thereby avoids any conflicts. In planning a given problem it is important to bear in mind the distinction between actual numbers or values involved in the calculation as compared with the operation (or instruction) to be performed on these values.

For example, in completing the calculation of "A" multiplied by "B" equals "C", the literal indication of the values involved are "A", "B" and "C" whereas the operation is multiplication. In a particular case the symbolic indications assume particular values such as "A" equal to 9, "B" equal to 6, in which case "C" would be 54. When writing the "Operation" column and the right half of the planning chart, the literal symbols are used in keeping track of factors and results rather than specific numbers themselves. Thus the sequence of operations is established for the general case of the problem. The specific numerical values corresponding to a particular case may then enter at the start of the problem and the results for this particular case be obtained. For a given problem, the sequence of operations remains invariant from one case to another. Only the numerical values for each of the parameters change from one case to the next. Thus, in planning a problem, the sequence of operations is divorced from the numerical values of a particular case. The CPC automatically distinguishes between two different types of numbers. Numbers which are instructions which tell the machine which operation to perform at what time, upon other numbers which are actual values involved in the solution of a particular case. To maintain this distinction the term "instruction" is used on the one hand and "factor" "value", or "result" is used on the other.

Having planned the sequence of individual arithmetic operations required to solve the general case and having assigned specific storage locations for each of the factors and results to be used in the calculation, it is necessary simply to code each individual step. The columns headed "Punch Into Instruction Cards" are used for this purpose. One card is punched for each line of the planning chart. Cards are numbered serially to make certain the operations are carried out in the proper sequence. The proper number for each line is recorded in the "Card No." column.

The "Channel A", "Channel B", and "Channel C", instructions are each two digit numbers indicating the storage location in which a particular factor or result is stored. Notice that this does not give the actual value of the factor itself but rather indicates the storage location number in which the value is stored. Remember that each individual arithmetic step of the problem is of the form "A" Operation "B" equals "C". The instructions simply tell the machine where the "A" and "B" factors are situated in storage and where to store the result "C".

The "Operation" instruction (not to be confused with the operation description in column one of the planning chart) is a numeric code which tells the CPC which arithmetic step is to be performed such as addition, subtraction, multiplication, division, square root, sine, hyperbolic cosine, etc.

The "Shift" instruction tells the CPC to multiply or divide by a predetermined power of 10 to align values and results properly.

"Special Control" is used to instruct the calculator when to print and when to punch result cards as well as perform other desired functions.

"Card Entry into Channel A" and "Card Entry into Channel B" are used for recording the actual numerical values of factors entering the calculation. Where the value is a constant which does not vary from one case to the next it may be entered in an instruction card. If the value is peculiar to a particular case it is entered into a "loading" card which is distinct from an instruction card. In a given problem the loading cards may precede the instruction cards or they may be interspersed among the instruction cards. The loading values are kept separate from the instruction cards so the same instruction cards may be used over and over again using a new set of loading cards to solve each particular case of the general problem.

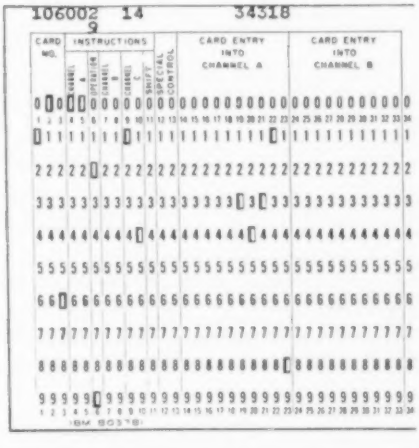
In a typical sequence type problem, such as the numerical integration of a differential equation, a relatively small number of loading cards, containing the numerical values of the starting conditions for a particular case, are loaded into storage registers of the CPC. These are followed by the instruction cards corresponding to the necessary arithmetic operations required to solve the general case. The general instruction cards control the CPC to calculate using the loaded values of the particular case. Thus one particular case is solved.

A different particular case is solved by replacing the loading cards of the previous case with the loading cards of the new case and using the same instruction cards. In this way, the instruction cards, corresponding to the arithmetic operations in the general case, are used repeatedly to solve several specific cases arising from different values of the parameters. As a rule, the number of instruction cards is considerably larger than the loading cards for one case. Having created the instruction cards they can be used repeatedly for solving different cases. The different cases may arise in a broad study at one time or they may occur as several individual instances over a long period of time. In this later situation the instruction cards are kept on file until needed.

As soon as the engineer or mathematician has completed the Planning Chart for the problem, the calculations can be completed by the operator of the CPC. The engineer or mathematician is thus relieved of the responsibility of carrying out the laborious details of the individual calculations. Usually he will make himself available to the CPC operator to answer any questions which may arise as the solution progresses. In certain types of problems the engineer or mathematician works in close cooperation with the machine operator as the pattern of future cases to be calculated may depend largely on the results of previous cases.

The completed Planning Chart is delivered to the CPC Computing Department. An IBM Key Punch operator transcribes the information from the section of the planning chart labeled "Punch Into Instruction Cards" into standard IBM cards. The standard IBM card is 7-3/8 inches long and 3-1/4 inches wide and contains 80 columns of information. In any one column, one alphabetic character or one numeral may be recorded. The information to be punched into the card from the planning chart requires only 33 columns. Therefore, only that portion of the IBM card which is actually used is shown in the following illustration.

One card is punched for each line of the planning chart. The illustration shows Card No. 106. The Channel A "00" instruction tells the CPC to read the 34318 value from the Card Entry into Channel A field of the card. Operation 2 is subtraction; 9 is



Card Form, Subtraction

summary punch the result of the calculation into a blank IBM card. The unpunched columns of Channel B signal the CPC to use the result of the previous calculation as the "B" value in subtraction. The result "C" of the subtraction is placed in storage 14 following the Channel C instruction.

Care is exercised throughout the entire machine calculation to assure the accuracy of input factor values and final results.

A complete printed record of all instructions, all values and all results of the calculations can be printed on the type 412 Accounting Machine during the calculation on the CPC if this is desired. The machine speed is 100 operations in one minute including time to make a complete printed record. If the engineer or mathematician is concerned with only selected intermediate results and final results then only those of interest can be printed. If results are desired in IBM card form these are recorded automatically in the type 527 Gang Summary Punch.

A wide variety of problems are being solved on the CPC as a general purpose computer. Among these are solution of simultaneous linear algebraic equations, formula evaluation, curve fitting, numerical solution of differential equations, matrix operations, and table look-up and automatic interpolation. As well as serving as a general purpose calculator the IBM Card-Programmed Electronic Calculator is in use as a special purpose computer solving elaborate problems in accounting and business control including preparation of financial statements, writing mortage loan amortization schedules, and computing manufacturing requirements.

Specific examples of problems in automotive research and design which have been solved on the IBM Card-Programmed Electronic Calculator are discussed in detail in this journal by Mr. F. F. Timpner, Dynamics Engineer, Chrysler Corporation.

APPLICATION OF THE CPC TO AUTOMOTIVE ENGINEERING PROBLEMS

FRED F. TIMPNER
Chrysler Corporation

The first part of this paper will describe some of the problems occurring in automotive engineering that we solve with IBM equipment. The second part will be devoted to the mechanics of programming a problem for the card programmed calculator.

About a year and a half ago we had a problem involving harmonic analysis of a cam contour that required the values of the first 33 harmonics. The best available harmonic analysis chart was a 72 ordinate schedule that would give the first 36 harmonics. However, the higher harmonics would be so inaccurate that we were faced with the necessity of going to a larger number of ordinates. Since the solving of a 72 ordinate schedule is so tedious and time consuming, the thought of a greater number of ordinates drove us to consider machine computation. We discussed the problem with Mr. Ruschman of our IBM section and worked out a procedure using his accounting machinery to work up to 360 ordinates or beyond if required and to any number of harmonics within the limits of the number of ordinates used.

The fourier series is represented by:

$$f(\theta) = A_0 + A_1 \operatorname{Sin} \theta + A_2 \operatorname{Sin} 2\theta + A_3 \operatorname{Sin} 3\theta \ldots$$
$$+ B_1 \operatorname{Cos} \theta + B_2 \operatorname{Cos} 2\theta + B_3 \operatorname{Cos} 3\theta \ldots$$

$$A_H = \frac{2}{N} \sum_{i=1}^{i=N} y_i \operatorname{Sin} H\theta$$

$$B_H = \frac{2}{N} \sum_{i=1}^{i=N} y_i \operatorname{Cos} H\theta$$

Where: H = Harmonic number

N = Number of ordinates

Y = Ordinate value

$\theta = \frac{2\pi}{N} i$

i = Ordinate number

We established a master deck for 360 ordinates for each harmonic from 1 through 40. The master deck contains the values of Sin $H\theta$ and Cos $H\theta$ with the necessary identification coding. From this master deck a duplicate working deck is reproduced. The values of the ordinates are punched into the working deck and then the products Y_i Sin $H\theta$ and Y_i Cos $H\theta$ are formed and summed by the multiplier. For anyone unacquainted with punched-card machinery this probably sounds like a lengthy process. Actually it only requires about 30 minutes per harmonic. The results that we got were so satisfactory that we began examining other problems as to their application to IBM equipment.

The next one that we attempted was engine main bearing loads. This problem was one that we had to consider every time a new engine was proposed or major changes made in existing ones. The method of solution formerly used was a graphical solution that required about

3-1/2 to 5 days for a skilled engineer. This graphical solution considered only the two cylinders on each side of the main bearing, and solved for points at 20^0 increments of crank angle. To have done more would have required a prohibitive amount of work. We devised a method of solving this problem on IBM equipment that permitted us to use the effect of all of the cylinders in the engine and use any increment of crank angle desired. It also recognized the firing order and bank angle. The time saving was considerable. With our old style accounting equipment we were able to work out a main bearing load in four hours time. We expect to halve this time with the CPC.

We have also done considerable work with engine cams and have used the IBM equipment to perform most of the calculations. Since these calculations are generally carried out to five places behind the decimal point and at 1^0 intervals, the work is quite tedious if done by hand. With the new equipment now available we have reduced to four hours the work that formerly required over a week. The machines have enabled us to do research on types of functions for cam contours that we would never consider using if we had to evaluate them by conventional calculating methods.

Another problem that we have solved was the output from a modified four bar linkage. This output was a function of two independent inputs. It was necessary that we know the output for various combinations of these two inputs. Normally we would have made graphical solutions that would have required a week or two. Instead, we set the problem up in mathematical form and fitted that to the new IBM equipment. In less than an hour we had all our answers.

In addition to these examples we have also solved problems in transmission design, brake design, and other research and design problems. We can even have the machine plot the data for us. Not only is the time saving considerable but the engineers who normally would have to perform these calculations are left free to carry on other work. Since the machine can work to a large number of places just as easily as to a few places, the answers come out more accurate — within the limitations of the data. But one great advantage is that we may now investigate a lot more avenues than we normally could. This will allow us to work out a number of solutions and then pick out the best one. Normally because of the time element you are committed to your first line of endeavor.

It might be of interest to describe the relation of the computing section to the engineer. The engineer supplies the problem and the mathematical knowledge required to make a numerical solution. The computing section then performs all of the operations of programming, card preparation, wiring, etc. necessary to securing the answers. The results, completely identified and printed on vellum, are returned to the engineer. The operation of these machines requires a specialist and it isn't practical to make such a specialist out of the engineers. We have had excellent results from this method.

For the remainder of this talk I will show you how we would actually set up a problem for the card programmed calculator. We will start with the boards already wired for general purpose operation. Our general purpose board will perform the following operations: add, subtract, multiply, divide, square root, e^x, e^{-x}, Sin x, Cos x, CosH x, SinH x, ARC Sin x, ARC Cos x, ARC TAN x, LN (1 + x). These boards have been wired for our machine and most problems can be fitted to them. If a problem should arise that would require functions beyond the capacity of this board, then a special board can be wired.

The programming for a sample problem is shown in Figure 2. The upper portion is the actual program sheet; the lower portion is for reference as to the program codes. The program sheet is divided into vertical columns corresponding to card columns. Each line represents a card. The first column headed "Operation" is for the convenience of the programmer who records the operation he is performing on each individual card. The second column is for the "Card Number" and the sub-heading 1, 2 and 3 refers to columns 1, 2 and 3 on the standard IBM card. (See Figure 1). The "Channel A" column (card columns 4 & 5) tells the machine

the location of factor "A". 00 causes the machine to read from "Card Entry Into Channel "A" (columns 17 thru 24). 11, 12 ... 21, 22 ... 91, 92 ... would instruct the machine to read from the appropriate storage units within the machine itself. (See right hand side of program sheet). The "Operation" column (card column 6 & 7) instructs the machine as to what mathematical operation is to be performed. (See the coding at the bottom of the page). "Channel B" column (card columns 8 & 9) has the same code as Channel A and gives the machine the location of Factor B. Channel C column has also the same coding as Channels A & B and informs the machine where to send the result "C". The shift column (card column 12) tells the machine what power of 10 to multiply the answer by as it sends the answer into storage. An 11 punch in column 13 instructs the machine to list (print) the answer. An 11 punch in column 14 instructs the summary punch to punch the answers in a new card. Card columns 17 thru 24 and card columns 27 thru 34 are for entering data into Channel A or Channel B.

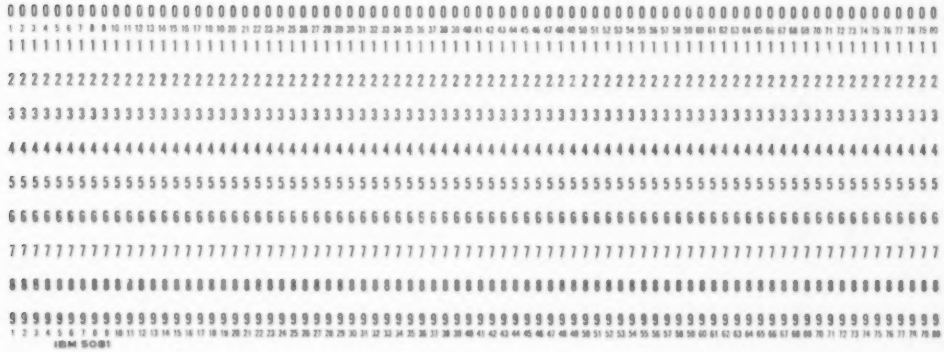


Figure 1

For the problem shown, $Y = e^{-At}[c_1 \cos wt + c_2 \sin wt] + B_3t^3 + B_2t^2 + B_1t + B_0$, the parameters A, B_0, B_1, B_2, B_3, C_1, C_2 and w must be entered into the machine before the calculation can begin. These data are added on the first 9 cards. On each card the factor is read from card entry into Channel A, added to zero which is read from card entry into Channel B and the sum sent into storage as designated in Channel C. The right hand section of the program sheet lists all of the storage untis so the programmer can keep track of what quantities are located in each unit. A quantity sent into the auxiliary storage registry is not available for use until two card cycles have passed, thus, "t" which is sent into storage register 21 on card 1 is not actually in the storage register until card 3 and cannot be used until card 4. Data sent into counters, however, need wait only one card cycle until it is available for use. The auxiliary storage registers automatically clear when a new factor is added into them, whereas the counter groups will only clear when specifically impulsed to clear and can be used to accumulate totals. All storage units keep track of the algebraic sign as well as the numerical quantity. After the data has been entered from the first nine cards designated as data cards, the equation is programmed on cards 10 thru 26. Each card performs one arithmetical operation until finally on card 26 the answer has been developed and is printed. This deck of instruction cards gives a solution for one value of "t". Card 24 increased the value of "t" in storage by Δ t so that the next deck of cards 10 thru 26 will develop the solution for the second value of "t" and increase "t" for the third solution; by stacking additional decks of instruction cards in the machine the caluclation can be carried forward as far as desired. Since each additional deck of instruction cards is identical with the first deck, only the data cards and first deck of instruction cards are punched by hand. The remaining decks of instruction cards are then automatically duplicated by an IBM reproducer.

CARD PROGRAMMED ELECTRONIC CALCULATOR PLANNING CHART

APPLICATION $Y = e^{At}(c_1 \cos wt + c_2 \sin wt) + B_3 t^3 + B_2 t^2 + B_1 t + B_0$

	OPERATION	CARD NO	CHANNEL A	OPERATION	CHANNEL B	CHANNEL C	SHIFT	LIST	SUMMARY	CARD ENTRY INTO CHANNEL A	CARD ENTRY INTO CHANNEL B
		1 2 3	4 5	6 7	8 9	10 11	12	13	14 15 16	17 18 19 20 21 22 23 24 25 26 27 28 29 30 31 32 33 34	
DATA CARDS	ADD t	001	00	10	00	21	2			0	
	ADD A 10^{-1}	002	00	10	00	11	2			2	
	ADD C_1 10^{-1}	003	00	10	00	12	2			35	
	ADD W 10^{-2}	004	00	10	00	13	2			3	
	ADD C_2 10^{-1}	005	00	10	00	14	2			025 "	
	ADD B_3 10^{-2}	006	00	10	00	15	2			30	
	ADD B_2 10^{-2}	007	00	10	00	16	2			27 "	
	ADD B_1 10^{-2}	008	00	10	00	17	2			15	
	ADD B_0 10^{-1}	009	00	10	00	18	2			35 "	
INSTRUCTION CARDS	Wt 10^{-2}	010	13	20	21	22	2				
	COS Wt 10^{-1}	011		45	00	00	2				15707963
	C_1 COS Wt 10^{-1}	012	12	20		97	3				
	SIN Wt 10^{-1}	013	22	40	00	00	2				15707963
	C_2 SIN Wt 10^{-1}	014	42	20		97	3				
	At 10^{-1}	015	11	20	21	00	2				
	e^{-At} 10^{-1}	016		75	00	00	2				0
	e^{-At} (CTR 97) 10^{-2}	017		20	97	95	2				
	t^2	018	21	20	21	22	2				
	t^3	019	21	20		00	2				
	$B_3 t^3$ 10^{-2}	020	15	20		95	2				
	$B_2 t^2$ 10^{-2}	021	16	20	22	95	2				
	$B_1 t$ 10^{-2}	022	17	20	21	95	2				
	t IN CTR	023	21	10	00	96	2				
	$\Delta t + t$	024	00	10		21	2			01	
	B_0 10^{-2}	025	18	10	00	95	1				
	PRINT	026	00	10	00	00	2	"			

COUNTER GROUPS: 91 92 93 94 95 96 97

AUXILIARY STORAGE REGISTERS: 11 12 13 14 15 16 17 18 19 20 21 22 23 24 25 26 27 28

OPERATION	OPERATION CODE	NORMAL SHIFT CODE	RESULT AVAIL	CHANNEL B	OPERATION	OPERATION CODE	NORMAL SHIFT CODE	RESULT AVAIL	CHANNEL B
A + B	10	2	ABCD		$^1/_{10}$ A/B	36	1	ABCD	
$^1/_{10}$ [A + B]	11	1	ABCD		10 A/B	37	2	ABCD	
A − B	15	2	ABCD		$^1/_{10}$ SIN $^1/_{10}$ A	40	2	ABC	$\pi/_{20}$
$^1/_{10}$ [A − B]	16	1	ABCD		$^1/_{10}$ COS $^1/_{10}$ A	45	2	ABC	$\pi/_{20}$
AB	20	2	ABCD		$^1/_{10}$ ARC SIN $^1/_{10}$ A	50	2	ABC	$\sqrt{2}/_{20}$
$^1/_{10}$ AB	21	1	ABCD		$^1/_{10}$ ARC COS $^1/_{10}$ A	55	2	ABC	$\sqrt{2}/_{20}$
10 AB	22	2	ABCD		$^1/_{10}$ SINH $^1/_{10}$ A	60	2	ABC	0
−AB	25	2	ABCD		$^1/_{10}$ COSH $^1/_{10}$ A	65	2	ABC	0
$^{-1}/_{10}$ AB	26	1	ABCD		$^1/_{10}$ e $^1/_{10}$ A	70	2	ABC	0
−10 AB	27	2	ABCD		$^1/_{10}$ e$^{-1}/_{10}$ A	75	2	ABC	0
$\sqrt{A}$	30	2	ABCD		$^1/_{10}$ LN $^1/_{10}$(1 − B)	80	2	ABC	$^2/_{10}$
$^1/_{10}$ $\sqrt{A}$	31	1	ABCD		$^1/_{10}$ ARC TANH $^1/_{10}$ A	85	2	ABC	0
A / B	35	2	ABCD		$^1/_{10}$ ARC TAN $^1/_{10}$ A	88	2	ABC	$^1/_{10}$

" PUNCH OVER UNIT POSITION ADDS PREVIOUS RESULT ON ARITHMETIC OPERATIONS.

SHIFT CODE

0	10^{-2}
1	10^{-1}
2	10^{0}
3	10^{1}
4	10^{2}
5	10^{3}
6	10^{4}

CHANNEL C: 9 IN TENS POSITION - CRT RI PLUS.
" IN TENS POSITION - CRT RI MINUS.

Figure 2.

This particular board is wired for 8 digit arithmetic, i.e., an 8 digit number times an 8 digit number gives an 8 digit product. The machine recognizes factors "A", "B" and "C" to be less than one, so the programmer must multiply his data by the appropriate powers of 10 to fit into the machine and must keep track of this factor throughout the calculation so that a number multiplied by 10^{-1} is not added to another number multiplied by 10^{-2}. For example: on card 12 and card 14 the product would have been multiplied by 10^{-2} but it was shifted up by a power of 10 before storing in counter 97. Then when counter 97 was read out and multiplied by e^{-At} 10^{-1} the product was 10^{-2} and could be added directly into counter 95 where all of the other entries were to be times 10^{-2}.

To solve this equation for different parameters only the data cards need be changed. Programming an equation is analagous to tooling up to manufacture a part. A lot of time and effort goes into producing the first part but the succeeding ones come quickly after that. By storing the cards (or program sheets) your "tooling" is preserved for any future solutions required.

Everyone in our organization who has come in contact with this equipment has been completely sold on it. Machine calculation is in its infancy but it is growing rapidly. In a few years it will be a necessity in all research or design groups.

BENDING OF CIRCULAR PLATE WITH ECCENTRIC HOLE

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1. Introduction

The purpose of this paper is to determine the stress and strain within a thin plate whose cross section consists of two circles, one interior to and eccentric to the other when that plate is subjected to a force acting perpendicular to the upper surface. The plate is assumed to be homogeneous and isotropic and it is assumed that there are no forces acting in the plane of the plate. It is well known that the equation governing the small deflections of such a plate is:

$$D\Delta\Delta w + P(x,y) = 0 \tag{1}$$

where P(x,y) is the load function per unit area,

$$D = \frac{2d^3}{3(1-\sigma^2)}$$

is the flexural rigidity of the plate, Δ is the two-dimensional Laplacian operator, and w(x,y) is the deflection of points originally on the middle plane of the plate. The bending moments and shearing stresses can be expressed directly in terms of the deflection w and hence the state of stress within the body is completely determined if a solution of equation (1) under suitable boundary conditions is known. We shall present a detailed solution when the load is uniform, but the method of solution readily extends to arbitrary load functions.

2. Bi-polar coordinates.

Because of the nature of the cross section of the plate, the ideal coordinates for the treatment of this problem are bipolar coordinates (α,β) which are defined by the conformal transformation.

$$z = x+iy = c\tanh\tfrac{1}{2}(\alpha+i\beta) = c\tanh\tfrac{1}{2}\zeta \tag{2}$$

The relationship between (x,y) and (α,β) is given by

$$x = \frac{1}{h}\sinh\alpha\,, \qquad y = \frac{1}{h}\sin\beta$$

where $h = (\cosh\alpha+\cos\beta)/c$ is the Jacobian of the transformation. This transformation maps the rectangular region

$$[\alpha_1 \le \alpha \le \alpha_0\,,\ -\pi \le \beta \le \pi]$$

in the ζ-plane onto the region between two eccentric circles in the z plane. The lines α =

constant map into two sets of circles about the poles $x = \pm c$, the poles themselves corresponding to $\alpha = \pm\infty$. The lines β = constant map into another set of circles with centers along the y-axis and orthogonal to the α circles. The real axis defined by $-c < x < c$ corresponds to $\beta = 0$ while the rest of the real axis corresponds to $\beta = \pi$. The plate considered here has a cross section bounded by the circles $\alpha = \alpha_0$, $\alpha = \alpha_1$ with $\alpha_0 > \alpha_1 > 0$ as shown in Fig. 1.

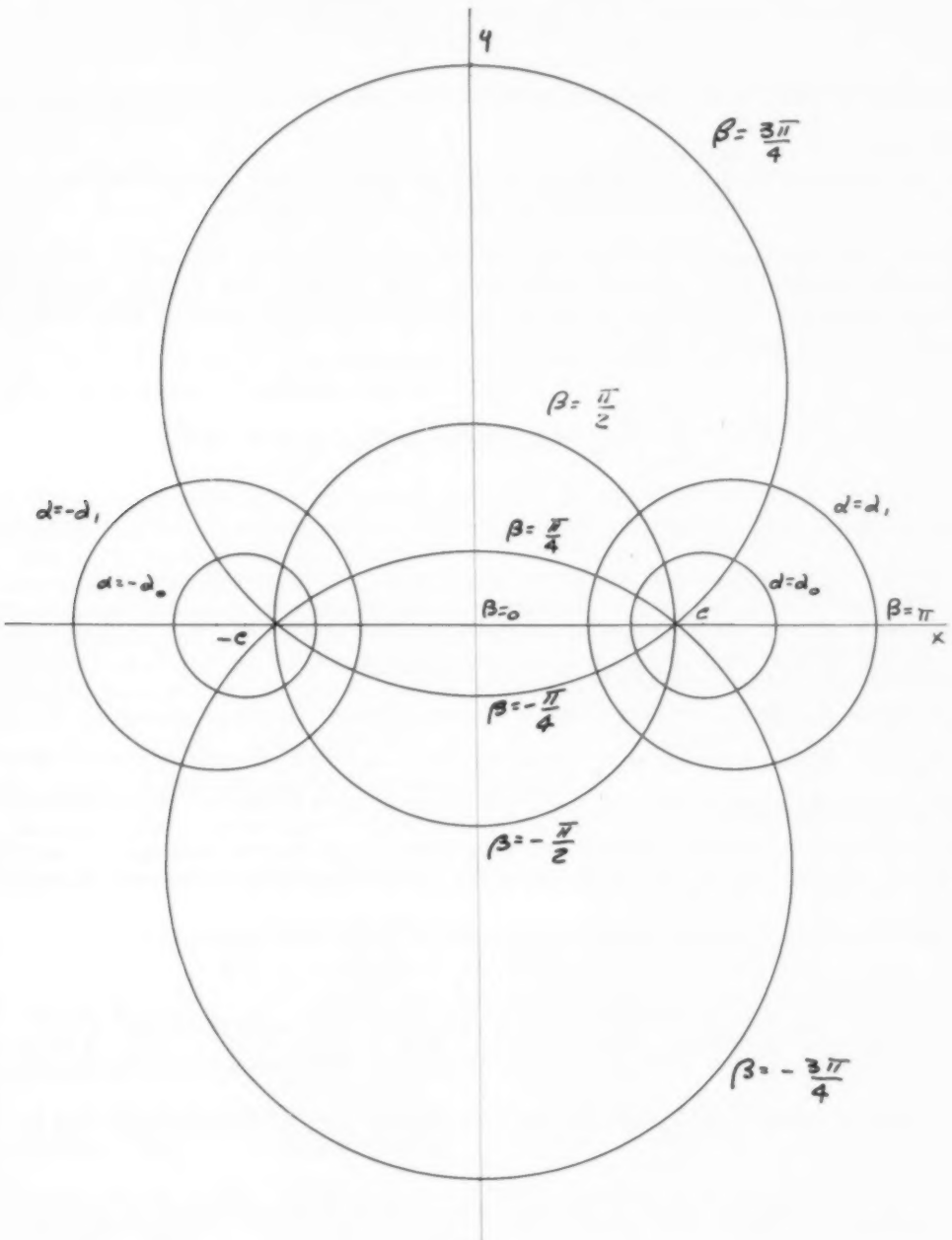

Figure 1.

Under the transformation (2) the Laplacian and biharmonic operators become

$$\Delta\omega = \frac{\partial^2\omega}{\partial x^2} + \frac{\partial^2\omega}{\partial y^2} = k^2\left(\frac{\partial^2\omega}{\partial\alpha^2} + \frac{\partial^2\omega}{\partial\beta^2}\right).$$

$$\Delta\Delta\omega = \left[k^2\left(\frac{\partial^2}{\partial\alpha^2} + \frac{\partial^2}{\partial\beta^2}\right)\right]\left[k^2\left(\frac{\partial^2\omega}{\partial\alpha^2} + \frac{\partial^2\omega}{\partial\beta^2}\right)\right] \tag{3}$$

Now if (h w) is taken as the dependent variable instead of w, the biharmonic equation becomes

$$\Delta\Delta\omega = k^3\left(\frac{\partial^4}{\partial\alpha^4} + \frac{2\partial^4}{\partial\alpha^2\partial\beta^2} + \frac{\partial^4}{\partial\beta^4} - \frac{2\partial^2}{\partial\alpha^2} + \frac{2\partial^2}{\partial\beta^2} + 1\right)(k\omega) = 0 \tag{4}$$

Thus we have the remarkable result that the biharmonic equation becomes a linear partial differential equation with constant coefficients. This equation may now be treated by the classical method of separation of variables to yield the following series of biharmonic functions which are of period 2π in β:

$$k\omega = \sum_{n=0}^{\infty}\left\{\phi_n(\alpha)\cos n\beta + \psi_n(\alpha)\sin n\beta\right\} \tag{5}$$

where the coefficients are defined as follows:

$$\phi_0(\alpha) = A_0\sinh\alpha + B_0\alpha\sinh\alpha + D_0\cosh\alpha + E_0\alpha\cosh\alpha$$

$$\psi_0(\alpha) = 0$$

$$\phi_1(\alpha) = A_1 + B_1\alpha + D_1\cosh 2\alpha + E_1\sinh 2\alpha$$

$$\psi_1(\alpha) = A_1' + B_1'\alpha + D_1'\cosh 2\alpha + E_1'\sinh 2\alpha$$

$$\phi_n(\alpha) = A_n\cosh(n+1)\alpha + B_n\cosh(n-1)\alpha + D_n\sinh(n+1)\alpha + E_n\sinh(n-1)\alpha$$

$$\psi_n(\alpha) = A_n'\cosh(n+1)\alpha + B_n'\cosh(n-1)\alpha + D_n'\sinh(n+1)\alpha + E_n'\sinh(n-1)\alpha$$

and A_n, B_n, D_n, E_n, A_n', B_n', D_n', E_n' ($n = 0, 1, 2, \ldots$) are eight sets of arbitrary constants.

The equations for the bending moments expressed in bipolar coordinates are:

$$M_\alpha = Dk^2\left\{\frac{\partial^2\omega}{\partial\alpha^2} + \sigma\frac{\partial^2\omega}{\partial\beta^2} + (1-\sigma)\frac{\sinh\alpha}{c_k}\frac{\partial\omega}{\partial\alpha} + (1-\sigma)\frac{\sin\beta}{c_k}\frac{\partial\omega}{\partial\beta}\right\}$$

$$M_\beta = -Dk^2\left\{\frac{\partial^2\omega}{\partial\beta^2} + \sigma\frac{\partial^2\omega}{\partial\alpha^2} - (1-\sigma)\frac{\sinh\alpha}{c_k}\frac{\partial\omega}{\partial\alpha} - (1-\sigma)\frac{\sin\beta}{c_k}\frac{\partial\omega}{\partial\beta}\right\}$$

$$M_{\alpha\beta} = -M_{\beta\alpha} = Dk^2\left\{-(1-\sigma)\left(\frac{\partial^2\omega}{\partial\alpha\partial\beta} - \frac{\sin\beta}{c_k}\frac{\partial\omega}{\partial\alpha} + \frac{\sinh\alpha}{c_k}\frac{\partial\omega}{\partial\beta}\right)\right\}$$

3. The boundary conditions.

The most common boundary conditions are those corresponding to clamped and simply supported edges or a mixture of the two. At a clamped edge both the deflection and its normal derivative vanish. Expressed in terms of hw these conditions along a clamped edge are

$$hw = \frac{\partial(hw)}{\partial \alpha} = 0 \tag{6}$$

Along a simply supported edge both the deflection and the moment about the edge vanish leading to the conditions

$$hw = 0$$
$$M_\alpha = 0 \tag{7}$$

4. Method of solution.

The method of solution proposed here is perfectly general and applies to an arbitrary load function. We seek a solution of the boundary value problem consisting of the plate equation (1) and either of the boundary conditions (6) or (7), in the form

$$hw = W_0 + W_1 \tag{8}$$

where W_0 denotes the series of biharmonic functions (5) and W_1/h is a particular solution of the plate equation (1). If a particular solution is known, we expand W_1 into a Fourier series in β. Now W_0 is itself a trigonometric series in β involving eight sets of arbitrary constants. The boundary conditions on (hw) provide eight equations for determining these eight sets of constants, these equations being obtained by equating to zero the coefficients of $\sin \eta\beta$, $\cos \eta\beta$. When the load is symmetrically distributed with respect to the X-axis, the deflection is an even function of β and it is sufficient to take only those terms in W_0, which involve the even functions $\cos \eta\beta$. This situation occurs, for instance, when the load is uniform, the case which will be discussed in detail in this paper.

5. The solution under a uniform load, clamped edges.

Let the plate under consideration be clamped at both the inner and outer edges and let it be subjected to a constant, normal load p per unit area. What is required then is a solution of the system:

$$D\,\Delta\Delta w + p = 0$$
$$hw = \frac{\partial(hw)}{\partial \alpha} = 0 \quad \text{when } \alpha = \alpha_0, \alpha = \alpha_1, \tag{9}$$

It is readily shown that a particular solution of the plate equation in (9) is given by

$$W_1 = \frac{A}{h} \qquad A = -\frac{pc^2}{16D} \tag{10}$$

From the definition

$$h = \frac{1}{c}(\cosh \alpha + \cos\beta)$$

it is evident that h is an even function of β and has a Fourier cosine series expansion

$$W_1 = a_0(\alpha) + \sum_{n=1}^{\infty} a_n(\alpha) \cos n\beta \tag{11}$$

The coefficients may be obtained by integration about the unit circle in the complex plane; they are:

$$a_0(\alpha) = -\frac{pc^2}{16D} \cdot \frac{1}{\sinh \alpha}$$

$$a_n(\alpha) = (-1)^{n+1} \frac{pc^2}{16D} \cdot \frac{2e^{-n\alpha}}{\sinh \alpha}, \quad n \geq 1 \tag{12}$$

For the biharmonic functions we take

$$w_0(\alpha, \beta) = \phi_0(\alpha) + \sum_{n=1}^{\infty} \phi_n(\alpha) \cos n\beta \tag{13}$$

where $\phi_n(\alpha)$ are defined in Section 2 and involve the arbitrary constants A_n, B_n, D_n, E_n, $(n = 0, 1, 2, \ldots)$. Substituting the expressions (11) and (13) for W_1 and W_0 respectively into the form (8) for hw we have

$$hw = \phi_0 + a_0 + \sum_{n=1}^{\infty} \left\{ \phi_n + a_n \right\} \cos n\beta \tag{14}$$

We apply now the boundary conditions in (9) and since the resulting equations must hold for all values of β we may equate to zero the coefficients of $\cos \eta\beta$ in each equation to obtain

$$\phi_0(\alpha_0) + a_0(\alpha_0) = 0$$
$$\phi_n(\alpha_0) + a_n(\alpha_0) = 0 \tag{15}$$

$$\phi_0(\alpha_1) + a_0(\alpha_1) = 0$$
$$\phi_n(\alpha_1) + a_n(\alpha_1) = 0 \tag{16}$$

$$\phi_0'(\alpha_0) + a_0'(\alpha_0) = 0$$
$$\phi_n'(\alpha_0) + a_n'(\alpha_0) = 0 \tag{17}$$

$$\phi_0'(\alpha_1) + a_0'(\alpha_1) = 0$$
$$\phi_n'(\alpha_1) + a_n'(\alpha_1) = 0 \tag{18}$$

Primes denote differentiation with respect to α. It can be shown that these equations have a unique solution for the constants A_n, B_n, D_n, E_n, $(n = 0, 1, 2, \ldots)$ which appear in $\phi_n(\alpha)$. Furthermore, the series solution (14) with these values of the arbitrary constants converges uniformly over the domain

$$\left[\alpha_1 \leq \alpha \leq \alpha_0, \quad -\pi \leq \beta \leq \pi\right]$$

and satisfies the system (9). Hence it represents the unique solution for the small deflections of a plate clamped at both edges. The proof of these matters is omitted for lack of space and

in any case is available elsewhere [1]. It may be of interest to exhibit the nature of the coefficients. If for convenience we use the notation

$$\rho_n = \sinh n\alpha_0, \quad c_n = \cosh n\alpha_0,$$

$$s_n = \sinh n\alpha_1, \quad C_n = \cosh n\alpha_1$$

the equations for A_n, B_n, D_n, E_n, $(n = 2, \ldots)$ are

$$A_n c_{n+1} + B_n c_{n-1} + D_n \rho_{n+1} + E_n \rho_{n-1} = -a_n(\alpha_0)$$
$$A_n C_{n+1} + B_n C_{n-1} + D_n s_{n+1} + E_n s_{n-1} = -a_n(\alpha_1)$$
$$(n+1)A_n \rho_{n+1} + (n-1)B_n \rho_{n-1} + (n+1)D_n c_{n+1} + (n-1)E_n c_{n-1} = -a_n'(\alpha_0)$$
$$(n+1)A_n s_{n+1} + (n-1)B_n s_{n-1} + (n+1)D_n C_{n+1} + (n-1)E_n C_{n-1} = -a_n'(\alpha_1)$$

We may solve this set of equations by Cramer's rule denoting the determinant of the coefficients by δ_n and the corresponding determinants for A_n, B_n, D_n, E_n, by δA_n, δB_n, δD_n, δE_n respectively. The results are:

$$\delta_n = 2\cosh 2n(\alpha_0 - \alpha_1) - 2n^2\cosh(\alpha_0 - \alpha_1) + 2(n^2-1)$$

$$\delta A_n = -a_n(\alpha_0)\left\{(n^2-1)c_{n+1} - (n^2-1)C_{n+1}\cosh(n-1)(\alpha_0-\alpha_1) - (n-1)^2 s_{n+1}\sinh(n-1)(\alpha_0-\alpha_1)\right\}$$
$$+ a_n(\alpha_1)\left\{-(n^2-1)C_{n+1} + (n^2-1)c_{n+1}\cosh(n-1)(\alpha_0-\alpha_1) - (n-1)^2\rho_{n+1}\sinh(n-1)(\alpha_0-\alpha_1)\right\}$$
$$- a_n'(\alpha_0)\left\{-(n-1)\rho_{n+1} + (n-1)s_{n+1}\cosh(n-1)(\alpha_0-\alpha_1) + (n+1)C_{n+1}\sinh(n-1)(\alpha_0-\alpha_1)\right\}$$
$$+ a_n'(\alpha_1)\left\{(n-1)s_{n+1} - (n-1)\rho_{n+1}\cosh(n-1)(\alpha_0-\alpha_1) + (n+1)C_{n+1}\sinh(n-1)(\alpha_0-\alpha_1)\right\}$$

$$\delta B_n = a_n(\alpha_0)\left\{-(n^2-1)c_{n-1} + (n^2-1)C_{n-1}\cosh(n+1)(\alpha_0-\alpha_1) + (n+1)^2 s_{n-1}\sinh(n+1)(\alpha_0-\alpha_1)\right\}$$
$$- a_n(\alpha_1)\left\{(n^2-1)C_{n-1} - (n^2-1)c_{n-1}\cosh(n+1)(\alpha_0-\alpha_1) + (n+1)^2 s_{n-1}\sinh(n+1)(\alpha_0-\alpha_1)\right\}$$
$$+ a_n'(\alpha_0)\left\{(n+1)\rho_{n-1} - (n+1)s_{n-1}\cosh(n+1)(\alpha_0-\alpha_1) - (n-1)C_{n-1}\sinh(n+1)(\alpha_0-\alpha_1)\right\}$$
$$- a_n'(\alpha_1)\left\{-(n+1)s_{n-1} + (n+1)\rho_{n-1}\cosh(n+1)(\alpha_0-\alpha_1) - (n-1)c_{n-1}\sinh(n+1)(\alpha_0-\alpha_1)\right\}$$

To obtain δD_n interchange c_{n+1} with s_{n+1}, and C_{n+1} with S_{n+1} in δA_n and change all signs. Similarly δE_n is obtained from δB_n by interchanging c_{n-1} with s_{n-1}, C_{n-1} with S_{n-1}, and changing all signs. These formulas apply only for $\eta \geq 2$, those for n = 0, 1 being slightly different in form. Experiments have been conducted which indicate that the solution given above is a very good approximation to the actual deflections of the plate considered here. The maximum deflection seems to occur at the point ($\beta = \pi$, $\alpha = \bar{\alpha}$) where $\bar{\alpha}$ is approximately midway between α_0 and α_1.

6. Extension to arbitrary load functions.

When the plate is subjected to an arbitrary normal load, the method given above can be applied provided that a particular solution of the plate equation can be obtained. The author has shown elsewhere (2) that a particular solution can always be obtained if the load function $P(\alpha,\beta)$ has a Taylor series expansion.

In other cases when the load function has certain specialized forms an exact solution of the plate equation for the case of clamped edges can readily be obtained. For example let

$$P(\alpha,\beta) = K h^3 \sinh(p\alpha)\cos(q\beta)$$

Where k is any given constant and p, q are integers but $p \neq (q \pm 1)$. If the plate is clamped at both edges a complete closed form solution is given by

$$h w = A \sinh(p\alpha)\cos(q\beta) + \phi_q(\alpha)\cos(q\beta)$$

where

$$\phi_q(\alpha) = A_q \cosh(q+1)\alpha + B_q \cosh(q-1)\alpha + D_q \sinh(q+1)\alpha + E_q \sinh(q-1)\alpha$$

The constant "A" is given by

$$A = -\frac{K}{D} \cdot \frac{1}{(p^2-q^2)^2 - 2q^2 - 2p^2 + 1}$$

and the constants A_q, B_q, D_q, E_q are determined uniquely by the boundary conditions.

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THE CONSTRUCTION AND USE OF NOMOGRAPHIC CHARTS

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Nomography (etymologically, "the science of drafting laws") is the name coined by Professor M. D'Ocagne (sometimes spelled d'Ocagne) of Paris for the science he created during the last two decades of the nineteenth century. It is the science of solving by graphical methods equations in more than two variables — in particular, that of finding from a given equation $\phi(x,y,z,\dots) = 0$ the value, or values, of one of the variables, say x, for a given set of values of the other variables without specifically setting up the equation $x = F(y,z,\dots)$.

Two approaches to the subject have been developed — a geometric approach based upon similar triangles, and an algebraic approach using the third order determinant. The writer has approached the subject over both routes over a period of several years, and has concluded that the determinant approach is easier both for the learner and for the teacher. This paper uses that approach exclusively.

1. Determinants.

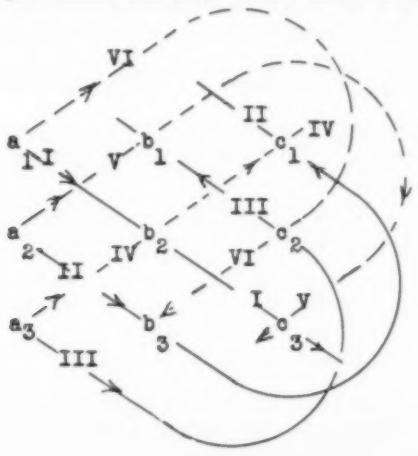
Let the array

$$(1) \qquad \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}$$

be the symbol for the quantity

$$(2) \qquad a_1b_2c_3 + a_2b_3c_1 + a_3b_1c_2 - a_3b_2c_1 - a_2b_1c_3 - a_1b_3c_2 \ .$$

Array (1), together with the operation of expansion by which (2) is derived from (1), is called a determinant of the third order, and the quantity (2) is the value of the determinant. The signs of the respective terms in (2) are determined by rules which are important in the formal study of determinants as a topic in mathematics. These rules may be found in any standard text in College Algebra, or in formal treatises on Matrices and Determinants. It suffices for our purpose, however, to give these rules by means of the diagram below:



The products of the elements lying on paths numbered I, II, and III are taken without change of sign, for, on path I, we find elements a_1, b_2, and c_3; on path II are a_2, b_3, and c_1; on III are a_3, b_1, and c_2. Products of elements lying on paths marked IV, V, and VI are taken with change of sign. To illustrate, the determinant

$$\begin{vmatrix} 2 & 4 & 5 \\ 3 & 2 & 7 \\ 6 & 1 & 8 \end{vmatrix} \quad 2x2x8 + 3x1x5 + 6x7x4 - 6x2x5 - 3x4x8 - 2x7x1 = 45 .$$

Similarly

$$\begin{vmatrix} 2 & -4 & 5 \\ 3 & 2 & -7 \\ -6 & 1 & -8 \end{vmatrix} = 2x2x(-8) + 3x1x5 + (-6)x(-7)x(-4) - (-6)x2x5 - 3x(-4)x(-8) - 2x(-7)x1 = -207 .$$

The reader should cultivate the habit of taking the elements in their order of their occurrence on the paths as they are followed in the direction of the arrow heads.

The second expansion just shown shows the necessity for extreme care in handling negative signs. Inaccuracy here is the source of much of the difficulty in handling the whole subject of Nomography.

A passing remark — determinants of second, fourth, fifth, ... orders exist and are very important in their appropriate fields in mathematics. It should be remarked also that the diagrammatic expansion shown above gives not the least intimation of the general method of expanding a determinant of any order. It is an accident — a happy one — that this method gives the same result as that given by the formal rules for expanding a determinant. It is a happy accident because it lends itself in particularly effective manner to Nomography. Another remark is that Nomography uses no determinant other than that of the third order.

Certain very important properties of determinants of all orders will be stated without proof, but will be illustrated. The interested reader will find formal proofs in places already indicated.

1. If two rows (horizontal) or two columns (vertical) of a determinant are interchanged, the numerical value of the determinant is unaffected, but the algebraic sign is reversed. Thus, where as we found that the determinant $\begin{vmatrix} 2 & 4 & 5 \\ 3 & 2 & 7 \\ 6 & 1 & 8 \end{vmatrix}$ had the value 45, the reader will find that each of the determinants $\begin{vmatrix} 2 & 5 & 4 \\ 3 & 7 & 2 \\ 6 & 8 & 1 \end{vmatrix}$, $\begin{vmatrix} 2 & 4 & 5 \\ 6 & 1 & 8 \\ 3 & 2 & 7 \end{vmatrix}$, and $\begin{vmatrix} 5 & 4 & 2 \\ 7 & 2 & 3 \\ 8 & 1 & 6 \end{vmatrix}$ has the value -45. Each of the last three determinants is formed by interchanging two rows, or two columns, of the original determinant.

Every determinant used in Nomography has the value zero — hence interchanging rows, or columns, has no effect whatever upon the value of such a determinant.

2. If the elements of any row, or of any column, be multiplied by any factor whatever, the value of the determinant is multiplied by that factor. The reader will verify, for example, that

$$\begin{vmatrix} 2 & 4 & 5 \\ 3 & 2 & 7 \\ 6m & m & 8m \end{vmatrix} = 45m .$$

Again, since the value of every determinant used in Nomography is zero, multiplying the elements of any row, or of any column, by any factor whatever has no effect upon the value of the determinant.

3. If the elements of any row, or of any column, be increased by adding to them respectively the products obtained by multiplying the corresponding elements of any other row, or of any other column, by any factor whatever, the value of the determinant is unchanged. For example, consider again the determinant

$$\begin{vmatrix} 2 & 4 & 5 \\ 3 & 2 & 7 \\ 6 & 1 & 8 \end{vmatrix} .$$

Now let the elements of the third column be increased by multiplying the elements of the second column by some factor m, and adding the results respectively to the elements of the third column, giving

$$\begin{vmatrix} 2 & 4 & 5 & 4m \\ 3 & 2 & 7 & 2m \\ 6 & 1 & 8 & m \end{vmatrix} .$$

The reader will verify that the values of the two determinants are the same.

It is necessary in Nomography to reverse the process of expansion. Thus, given the equation $2x + 3y = 4z$, or $2x + 3y - 4z = 0$, we write first

$$\begin{vmatrix} x & & \\ & 1 & \\ & & 2 \end{vmatrix}$$

which takes care of the term 2x. Then we write 3y into the array by writing

$$\begin{vmatrix} x & & 1 \\ y & 1 & \\ & 3 & 2 \end{vmatrix} .$$

Finally, we write -4z into the array by writing

$$\begin{vmatrix} x & & 1 \\ y & 1 & \\ 4z & 3 & 2 \end{vmatrix} .$$

Since there are no more terms to accommodate, we complete the determinant, and the equation, by writing

$$\begin{vmatrix} x & 0 & 1 \\ y & 1 & 0 \\ 4z & 3 & 2 \end{vmatrix} = 0 \quad .$$

Every problem in Nomography requires exactly what has been done above — it requires that an equation written in the conventional algebraic or trigonometric form be rewritten in determinant form. It is emphatically evident that skill in expanding determinants, and thorough familiarity with the diagrammatic scheme shown previously, are indispensable.

Two very important properties of the determinant shown above are to be noted:

(a) No horizontal row has in it more than one variable.

(b) All coefficients of variables are in the third row.

These two properties are built into every determinant used in Nomography, and skill in so doing is the first requisite for handling the whole subject. It is not always possible to put the algebraic equation into such a form. Even when it is possible it may take some manipulation and some ingenuity, as evidenced by the equation $xy - xz - yz = 0$. This can be written

$$\begin{vmatrix} x & 0 & 1 \\ 0 & y & 1 \\ z & z & 1 \end{vmatrix} = 0 \quad .$$

Here the first row contains x, and x only; the second contains y, and y only; the third row contains z, and z only. To be sure, there are two variable terms in the third row — but there is no variable other than z in the third row. The two variable terms in a given row may be different functions of the same variable — for example, the equation $x^2 + yz^2 = z^3$ can be written

$$\begin{vmatrix} x^2 & 0 & 1 \\ y & 1 & 0 \\ z^3 & z^2 & 1 \end{vmatrix} = 0 \quad .$$

It is still true that the third row contains one and only one variable.

For a final formal mathematical basis for the science we quote a theorem from Analytic Geometry:

"A necessary and sufficient condition that three points (a_1,b_1), (a_2,b_2) and (a_3,b_3) be collinear is that the value of the determinant

$$\begin{vmatrix} a_1 & b_1 & 1 \\ a_2 & b_2 & 1 \\ a_3 & b_3 & 1 \end{vmatrix}$$

shall be zero".

It follows from this theorem that if a determinant has its right hand column consisting of unit elements and if the value of the determinant is zero, then the elements of the first column are abscissas (or ordinates) and the elements of the second column are the ordinates (or abscissas) of three collinear points.

2. Parallel Axis Charts.

$$f_1(x) + f_2(y) = f_3(z) \ .$$

The given equation can be written

$$\begin{vmatrix} f_1 & 0 & 1 \\ f_2 & 1 & 0 \\ f_3 & 1 & 1 \end{vmatrix} = 0 \quad .$$

Adding the second column, termwise, to the third column (see property 3 of determinants) gives

$$\begin{vmatrix} f_1 & 0 & 1 \\ f_2 & 1 & 1 \\ f_3 & 1 & 2 \end{vmatrix} = 0 \quad .$$

Dividing the elements of the third row by 2 gives

$$\begin{vmatrix} f_1 & 0 & 1 \\ f_2 & 1 & 1 \\ \frac{1}{2}f_3 & \frac{1}{2} & 1 \end{vmatrix} = .0 \quad .$$

This determinant has the form quoted in the theorem mentioned from Analytic Geometry. In practice, however, some modification may be, and usually is, advisable. To illustrate, consider $2x + 3y = 4z$, or

$$\begin{vmatrix} x & 0 & 1 \\ y & 1 & 0 \\ 4z & 3 & 2 \end{vmatrix} = 0 \quad .$$

Adding the second column to the third and then dividing the elements of the third row by 5 gives

$$\begin{vmatrix} x & 0 & 1 \\ y & 1 & 1 \\ \frac{4}{5}z & \frac{3}{5} & 1 \end{vmatrix} = 0 \quad .$$

The first of these two forms using the determinant is called the basic form of the equation, and the second form is called the working form, for it is from the last form that we construct the chart. Since the third column consists of unit elements, we may take the elements of the first column as ordinates of collinear points and the elements of the second column as the abscissas of the same collinear points.

It is noted that every point associated with a value of x (first row) has the abscissa zero (second element of the first row). Then all points associated with x lie on a line (Fig. 1). At some convenient distance from this line a parallel line is drawn, and this distance between parallels will be called the unit of abscissas. Then all points associated with y lie on this second line. All points associated with z lie on a line parallel to the other two, and three fifths of the way from the x-axis to the y-axis.

The elements of the first column represent ordinates. The coefficients of x and y in the working form are equal, and the x- and y- axes will be graduated with the same scale. The coefficient of z is $\frac{4}{5}$, whence the scale used on the z-axis is $\frac{4}{5}$ that used on the other two axes. This suggests some such scale as $\frac{5}{16}$ inch per unit on the x- and y- axes, and $\frac{4}{16}$ inch per unit on the z-axis. The writer uses a six-scale engineers' boxwood scale, with scales of 10, 20, 30, 40, 50, and 60 to the inch. This makes relative scales very easy to pick out.

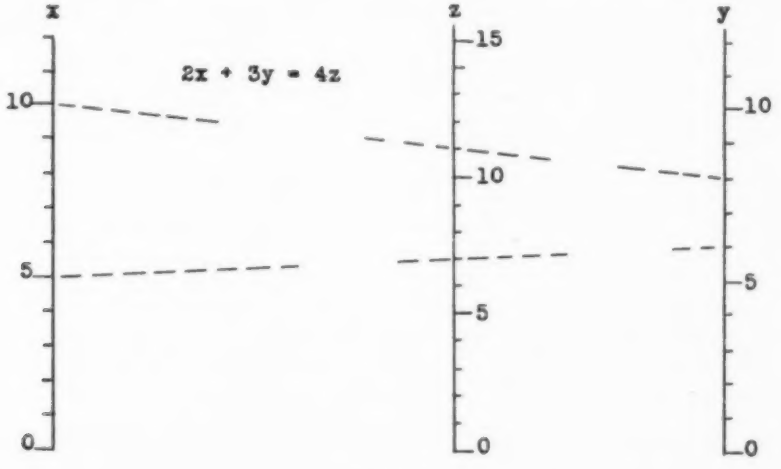


Figure 1

Since x = 0, y = 0, z = 0 is a solution of the equation, it follows from the theory we have shown that these three points must be collinear. Hence three collinear points are labeled 0, 0, and 0 on the three axes respectively. From these points the axes are graduated with the scales described. The accuracy of the chart is easily tested by drawing any line (called an isopleth) across the chart and testing the three points thus cut out to see if they satisfy the equation. The chart shows, for example, that z = 11 when x = 10, y = 8, and that y = 6 when x = 5, z = 7.

For the equation 2x + 3y = 4z + k it is necessary to find one solution. For example, for 2x +

$3y = 4z + 41$, at least one solution is $x = 4$, $y = 7$, $z = -3$. The equation can then be written $2(x-4) + 3(y-7) = 4(z\ 3)$, the working form of which is

$$\begin{vmatrix} x-4 & 0 & 1 \\ y-7 & 1 & 1 \\ \frac{4}{5}(z+3) & \frac{3}{5} & 1 \end{vmatrix} = 0 \quad .$$

The relative positions of the three axes, and the graduations are exactly the same as before. A basic set of collinear points is labeled 4, 7, and 3 respectively (Fig. 2). The reader will easily check by some sample solutions.

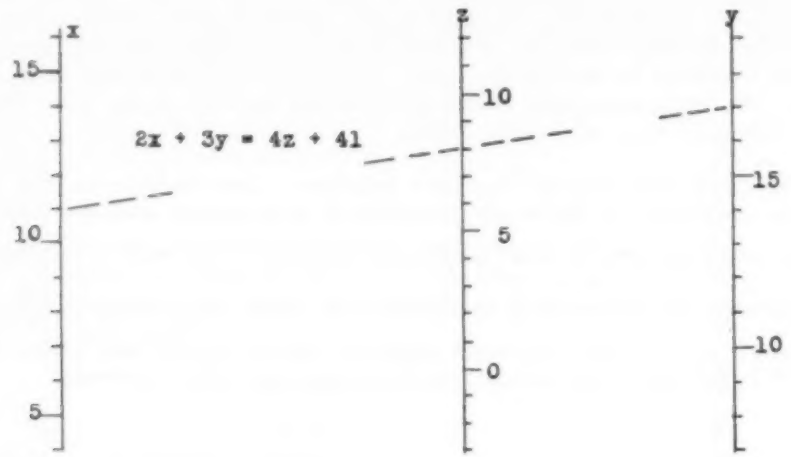


Figure 2

Our general equation can be written

$$m_1 f_1(x) + m_2 f_2(y) = m_3 f_3(z)$$

where m_1, m_2, and m_3 are numerical coefficients, called moduli, fitting the exigencies of the particular problem. In the two examples just shown it was assumed that the ranges of the variables, at least for two of them, are approximately the same. This is often not true in practice — the deflection of a beam loaded with weight measured in thousands of pounds may be measured in thousandths, or at most hundredths, of an inch. Where ranges are markedly different it becomes necessary or expedient to introduce scale changes, called moduli. The introduction of such moduli will now be discussed.

Consider again the equation $2x + 3y = 4z$, but let it be assumed that the problem arises in an environment prescribing a range of 0 to 20 for x and 0 to 50 for y, and let it be assumed also that a maximum limit on axis length is imposed by some consideration — by the size of a filing cabinet. The writer has imposed on himself in this paper a maximum axis length of 6 inches. Then the x-axis will be graduated 4 units to the inch, the y-axis 10 units to the inch. Our equation is then written

$$8\left(\frac{x}{4}\right) + 30\left(\frac{y}{10}\right) = 4z, \text{ giving the working form}$$

$$\begin{vmatrix} \frac{x}{4} & 0 & 1 \\ \frac{y}{10} & 1 & 1 \\ \frac{z}{9.5} & \frac{15}{19} & 1 \end{vmatrix} = 0 \quad .$$

The x-axis is graduated 4 units to the inch, Fig. 3, the y-axis 10 units to the inch, the z-axis 9.5 units to the inch. The abscissa of the z-axis is $\frac{15}{19}$ (the writer used $\frac{19}{4}$ inches and $\frac{15}{4}$ inches respectively). The scale on the z-axis, 9.5 to the inch, is not easily subdivided. To subdivide without numerical calculation, note that $z = \frac{1}{2}x + \frac{3}{4}y$. Select some value of y, say $y = 0$, whence $z = \frac{1}{2}x$. Lines from $y = 0$ to $x = 2,4,6,\ldots$ then cut out on the z-axis the points $x = 1,2,3,\ldots$ Similarly, if $x = 0$, then $x = \frac{3}{4}y$, and lines from $x = 0$ to $y = 20$ and $y = 40$ cut out $z = 15$ and $z = 30$ respectively. By choosing other values of x and of y, the z axis may be graduated as closely as desired.

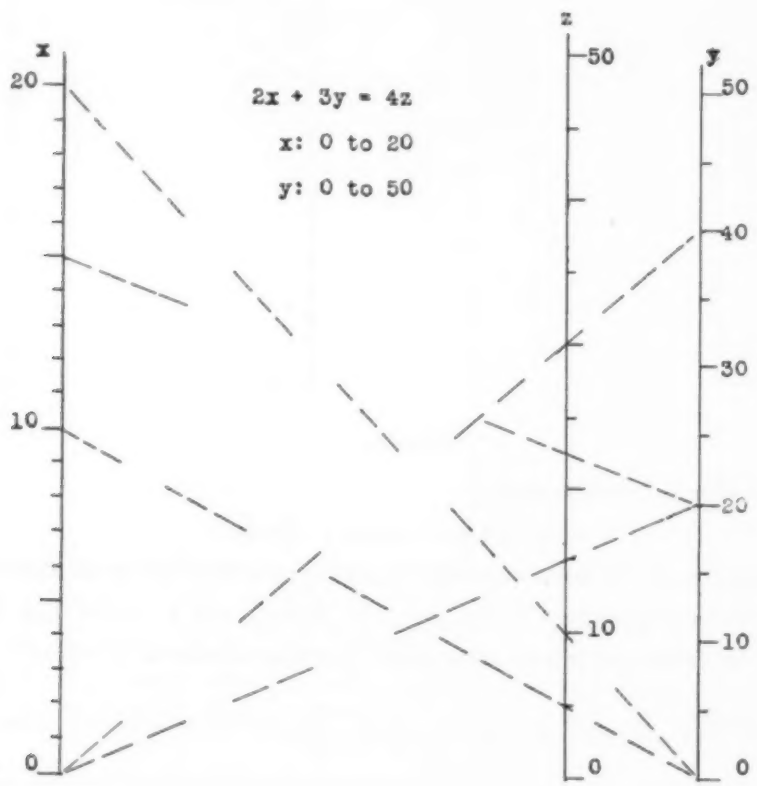


Figure 3

Certain equations not in the form of this section may be put into this form by algebraic manipulation. For Example, $x^a y^b = z^c$ may be written

$$a \log x + b \log y = c \log z,$$

which gives the working form

$$\begin{vmatrix} \log x & 0 & 1 \\ \log y & 1 & 1 \\ \frac{c}{a+b}\log z & \frac{b}{a+b} & 1 \end{vmatrix} = 0 \quad .$$

The left and right axes are graduated with the conventional log scale — a sheet of semilog paper is most convenient here — and the intermediate axis is graduated with a scale $\frac{c}{a+b}$ times the scale used on the x- and y-axes. Figure 4 is the chart of $x^2 y^3 = z^4$, with assumed equal ranges of x and y, 1 to 10.

Figure 4

The equation $x^a y^b = kz^c$ can be written

$$a \log x + b \log y = c \left(\log z + \frac{1}{c} \log k\right) \quad .$$

The relative positions of the three axes (Fig. 5) and the relative scale graduations are exactly as in Fig. 4. The only difference is that the point aligned with x = 1, y = 1 is the point $-\frac{1}{4}$ log k, which is easily computed from a log table. The figure is for $x^2 y^3 = 8.5\, z^4$.

Consider the equation

$$x^{2.5} y^{3.7} = z^{4.7}$$

which ranges as follows:

x: 200 to 300,000
y: 10 to 150 .

Logarithmically, these ranges are equivalent to 1 to 150 for x and 1 to 15 for y. Since $150 = 15^{1.8}$, approximately, the scale on the x-axis is $\frac{10}{18}$ that on the y-axis. Since log 15 1.17609 = $\frac{7}{6}$ approximately, and since the maximum axis length is to be 6 inches, the scale used will be a scale $6 \div \frac{7}{6} = \frac{36}{7}$ inches long. Since this is a rather awkward fraction, the linear intervals corresponding to the logs of the various values of the variables are found by numerical computation — a slide rule and a log table. This is probably more accurate than trying to modify

the scales geometrically, expecially when the scale is small enough to make drafting errors or even drafting tolerances relatively rather large. The equation is rewritten

$$4.5\left(\frac{5}{9}\log x\right) + 3.7 \log y = 4.7 \log z \ .$$

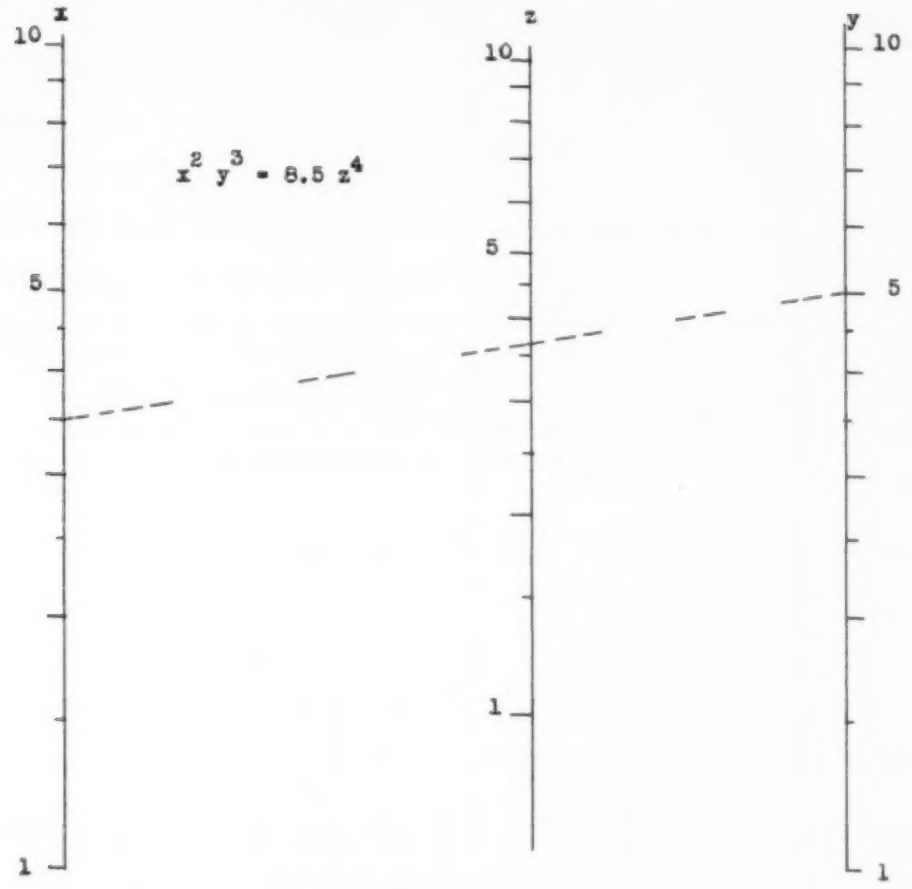


Figure 5

A little inspection shows that the abscissa of the z-axis is $\frac{37}{82}$ (Fig. 6). The writer used $\frac{164}{30}$ inches and $\frac{74}{30}$ inches respectively for the abscissas of the y- and the z- axes. To find a starting place on the z-axis, select a pair of values of x and z, say x = 1000, z = 1000. Then y = 60.88, and this point is easily spotted on the y-axis. Then the line joining x = 1000 to y = 60.88 gives the initial point z = 1000 from which the graduation of the z-axis is easily extended. An isopleth shows a specimen solution.

The reader will easily devise methods of using similar triangles for modifying scales. This is not, however, the similar triangle approach to Nomography which has been mentioned early in the paper.

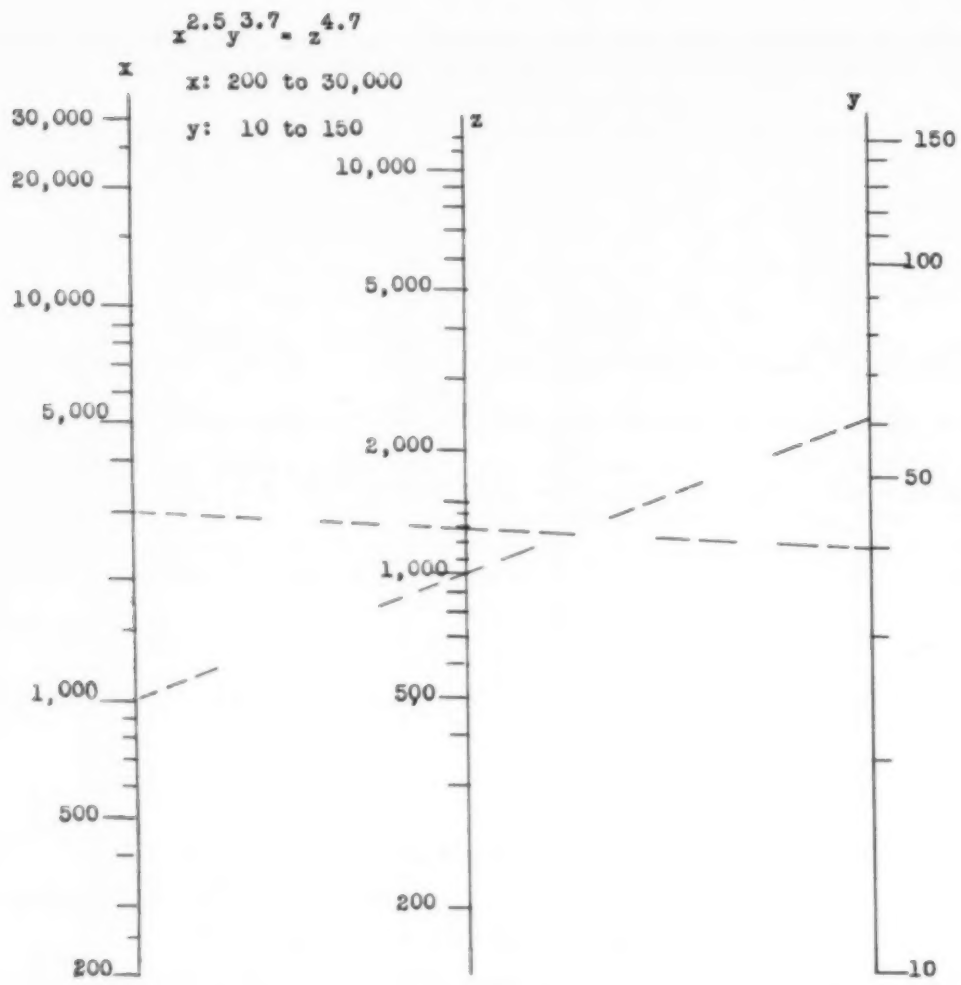


Figure 6

3. Equations In More Than Three Variables

An equation of the form

$$f_1(x) + f_2(y) + f_3(z) = f_4(w)$$

can be handled by introducing the auxiliary variable t in the pair of equations

$$f_1(x) + f_2(y) = f(t), \quad f(t) + f_3(z) = f_4(w) \ .$$

The first of these two equations gives a parallel axis chart with t as the intermediate axis; the second equation gives a parallel axis chart with t as one of the boundary axes. It may be possible, and usually is, to use moduli in such a way as to use the same t axis for both charts — that is, that the same scale is used for the t axis in both charts. We illustrate by the equation

$$2x + 3y + 4z = 7w \ ,$$

which is equivalent to the pair

$$2x + 3y = kt, \quad kt + 4z = 7w \ .$$

The first of these equations gives the working form

$$\begin{vmatrix} x & 0 & 1 \\ y & 1 & 1 \\ \frac{k}{5}t & \frac{3}{5} & 1 \end{vmatrix} = 0$$

and the second gives

$$\begin{vmatrix} t & 0 & 1 \\ z & 1 & 1 \\ \frac{7}{4+k}w & \frac{4}{4+k} & 1 \end{vmatrix} = 0 \quad .$$

For both charts to use the same t scale, it must be that k = 5. The t- axis is $\frac{3}{5}$ of the way from the x- axis to the y- axis, and the w- axis is $\frac{4}{9}$ of the way from the t- axis to the z- axis. (Fig. 7). There is no implied comparative size of the interval between the x- and y- axis and the interval between the t- and the w- axis. It is necessary only that the relative positions of the x, y, and t axes be as stated, and that the relative positions of the t, z, and w axes be as stated. It is not even necessary that the direction from the x to the y axis be the same as that from the t axis to the z axis — it is often economical of space to reverse these two directions. Fig. 8 shows the same equation with the reversed directions.

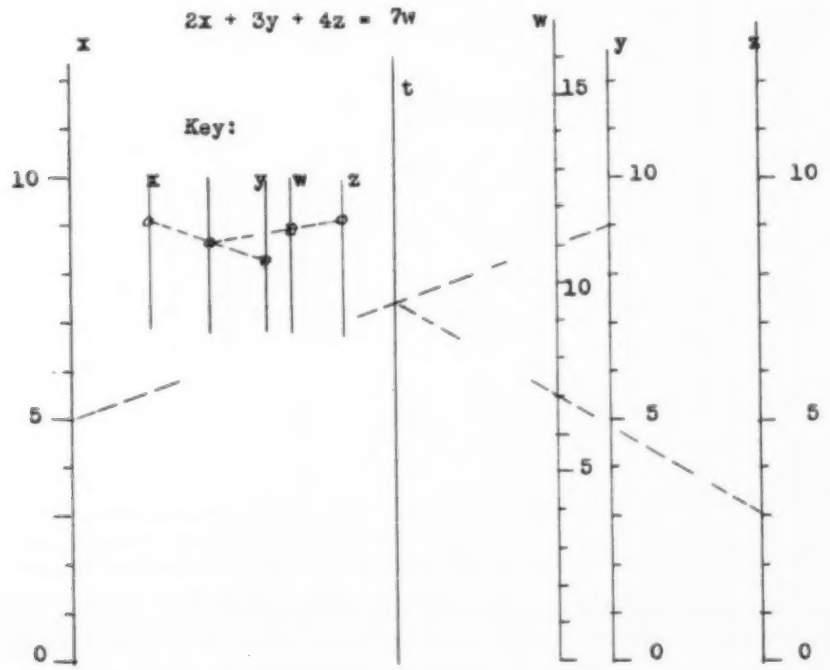
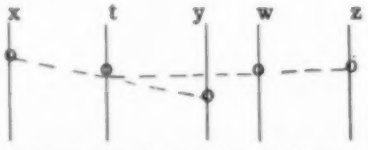


Figure 7

A chart of more than three variables must always be accompanied by a "key" to show its use. The key may be written in the form

KEY:
x--y--t
t--z--w

or it may be shown by the diagram



Each of the forms shows that values of x,y, and t are aligned, and that values of t, z, and w are aligned. Both methods are widely used, and there seems to be little ground for preference for either. Some form of key must be shown, however, to prevent troublesome or hopeless confusion.

The t axis above served no purpose other than that of a "bridge" to carry the result of the x and y computations, and it is therefore not graduated. If there is some reason for wanting to know partial sums it should be graduated.

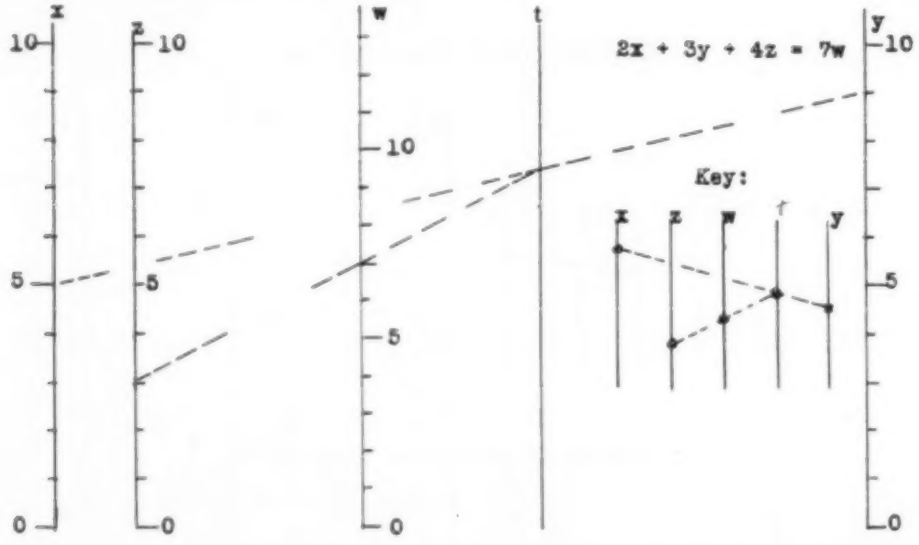


Figure 8

Now consider again the equation $2x + 3y = 4z$. The scale modification shown in the discussion of that equation can be avoided entirely by using the principle of similar triangles. In Fig. 9 let the abscissa of the line $M'N'$ be $\frac{b}{a+b}$ and let the abscissa of MN be $\frac{b}{c}$. Then the abscissa of MN is $\frac{a+b}{c}$ times the abscissa of $M'N'$, and the length of the segment MN is $\frac{a+b}{c}$ times the segment $M'N'$. But the scale on M N is $\frac{c}{a+b}$ times the scale on the x- axis, whence the scale on MN is the same scale as that used on the x- axis. To illustrate, Fig. 10 places the Z' axis $\frac{3}{5}$

of the way from the x- to the y- axis. Now let the abscissa of the line marked z be $\frac{3}{4}$ $(=\frac{b}{c})$ of the way from the x- axis to the y- axis. Then the segment M'N', which is on a $\frac{4}{5}$ scale, is magnified to MN, which is on a $\frac{1}{1}$ scale. Fig. 11 is the chart of $2x + 3y = 4z$ using this magnified scale. The auxiliary axis is not graduated or named. The key shows the use of the chart.

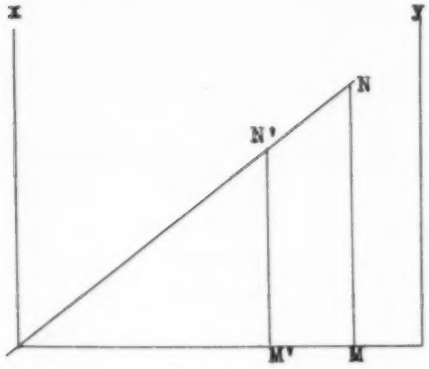


Figure 9

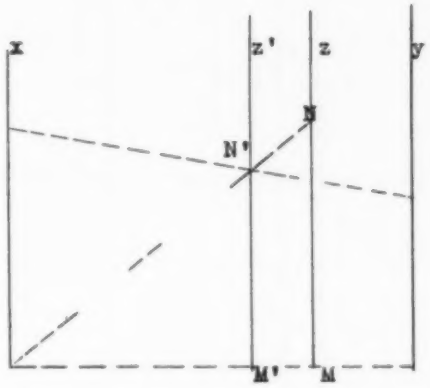


Figure 10

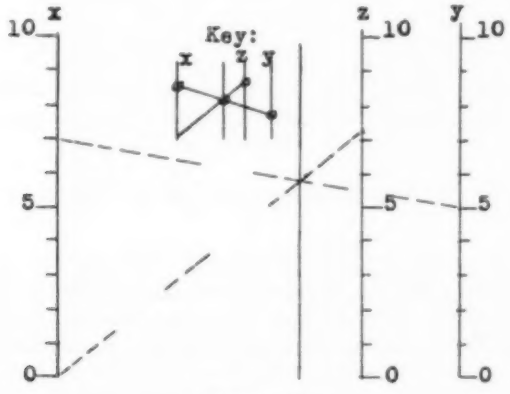


Figure 11

In the writer's opinion the advantages accruing from avoiding change of scale are more illusory than real. Every problem so charted requires an additional isopleth — which means additional time and one more chance for error. Modifying the scale is rarely difficult, and if the chart is to be used repeatedly — and it is only under such conditions that construction of a chart is economical — the time required to modify scales is more than made up in the time saved in using the chart.

An extended sum such as

$$f_1(x) + f_2(y) + f_3(z) + f_4(w) + \ldots\ldots = f(q)$$

can be written as a sequence of equations

$$f_1(x) + f_2(y) = F(t), \quad F(t) + f_3(z) = G(r),$$
$$G(r) + f_4(w) = H(s), \ldots.$$

For example, $2x + 3y + 4z + 10u + 8v = 11p$ is written as the sequence $2x + 3y = 5t$, $5t + 4z = 9r$, $9r + 10u = 19s$, $19s + 8v = 11p$. The chart is Fig. 12. Details of the construction are left to the reader.

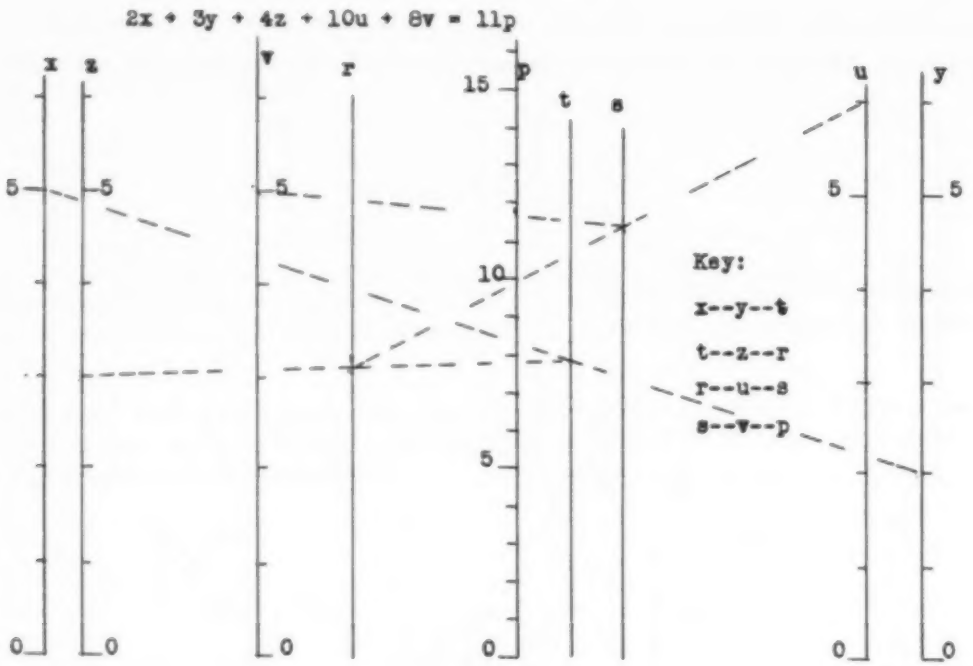


Figure 12

4. The N- Chart, or the Z- Chart.

$$f_1(x) \times f_2(y) = f_3(z) \quad .$$

Equations of this form can often be handled as in the preceding section — by taking logs of both sides, as was done there. Occasionally, however, this may be inconvenient or quite difficult. If any one of the functions involves additions or subtractions, for example, the use of logs is not convenient. Again, there may be particular reasons for preferring natural scales. This section discusses charting such equations by using natural scales.

The equation may be written in the basic form

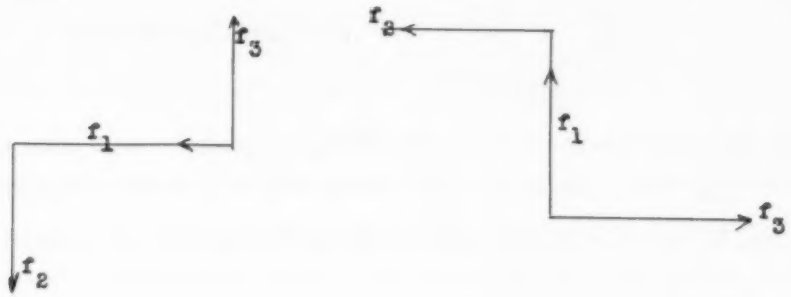
$$\begin{vmatrix} f_3 & 0 & 1 \\ -f_2 & 1 & 0 \\ 0 & f_1 & 1 \end{vmatrix} = 0$$

the working form of which is

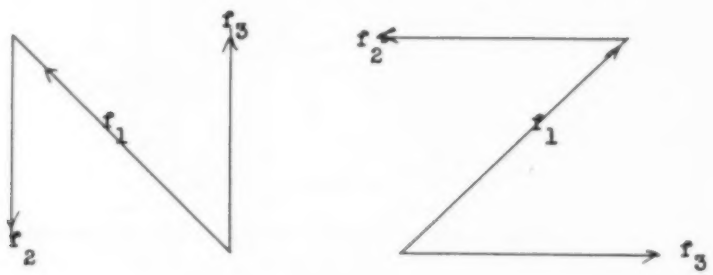
$$\begin{vmatrix} f_3 & 0 & 1 \\ -f_2 & 1 & 1 \\ 0 & \frac{f_1}{f_1+1} & 1 \end{vmatrix} = 0 \quad .$$

Of course f_2 and f_1 could have been interchanged. The working form shows that f_3 is charted on the left axis, f_2 on the right axis, and that the two axes extend in opposite directions from

the axis of zero ordinates. The line of zero ordinates carries f_1. The graph then appears in either of the forms



These graphs are both very wasteful of space – hence the chart is "sheared" to one of the forms



which brings the whole chart into convenient space. The names given to the chart, as in the heading of this section, result from the shape of the final graph.

To illustrate, consider

$$Z = \frac{BH^2}{6} \quad ,$$

the formula for the section modulus Z of a beam B inches wide and H inches high. For definiteness, consider beams up to 16 inches wide and 30 inches deep. Then the maximum value of Z is 2400. The basic form is

$$\begin{vmatrix} Z & 0 & 1 \\ -H^2 & 1 & 0 \\ 0 & B & 6 \end{vmatrix} = 0 \quad .$$

The working form is

$$\begin{vmatrix} Z & 0 & 1 \\ -H^2 & 1 & 1 \\ 0 & \frac{B}{B+6} & 1 \end{vmatrix} = 0 \quad .$$

This working form shows that the two parallel axes carry the variables Z and H. Since H^2 reaches 900, and since the maximum axis length is to be 6 inches, the scales on the Z and H axes respectively 400 units to the inch and 150 units to the inch. Then the working form will be rewritten

$$\begin{vmatrix} \frac{Z}{400} & 0 & 1 \\ -\frac{H^2}{150} & 1 & 1 \\ 0 & \frac{B}{B+16} & 1 \end{vmatrix} = 0 \quad . \quad \text{(The reader should verify).}$$

Graduating the Z and H axes offers no difficulites. To graduate the B axis, note that if $H^2 = 600$, then $B = \frac{Z}{100}$. Hence joining $H = \sqrt{600}$ to any value of Z locates the corresponding point $B = \frac{Z}{100}$. Fig. 13, the chart of this equation, shows B = 5 and B= 16 located in this manner.

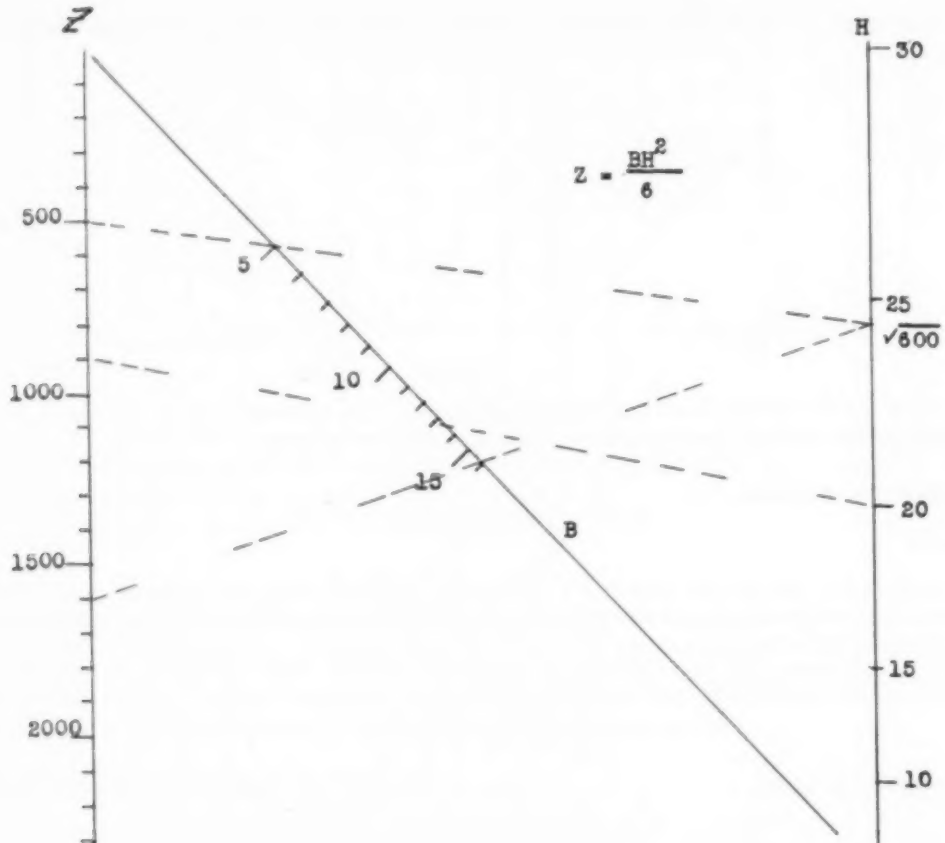


Figure 13

It is by now very evident that proper choice of moduli is essential, and is a large part of the whole process. Every problem presents its own peculiar difficulites, but some "trick" like that of graduating the B axis above is usually possible. It is worth spending some time hunting for such devices.

5. The Double N- , or the Box Chart

$$f_1(x) \times f_2(y) = f_3(z) \times f_4(w) \quad .$$

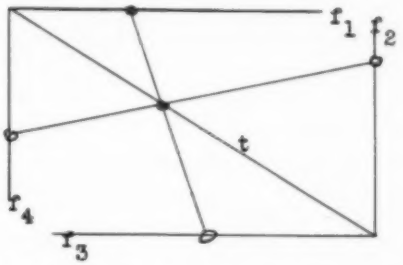
An equation of this kind can be handled in a manner very similar to that used to extend the parallel axis chart to more than three variables. Writing the equation

$$\frac{f_1}{f_3} = \frac{f_4}{f_2} = t \quad ,$$

we have the sequence

$$f_1 = t x f_3 \quad , \quad f_4 = t x f_2 \quad .$$

Again, as in the earlier extension, if moduli be so chosen that the t axis appears identically in the two working forms, then the two charts can be combined into the single chart



with the t axis ungraduated, as in the earlier extension. An example is

$$D = \frac{9\,W\,L^3}{E\,I} \quad ,$$

where D is the deflection in inches of a beam L feet long, loaded with W pounds at its mid-point, the ends of the beam being imbedded at both ends; E is the coefficient of elasticity of the beam material, I is the moment of inertia of the cross section of the beam with respect to the neutral axis. We shall consider a steel beam, with $E = 3 \times 10^7$. Writing the original equation

$$\frac{D}{L^3} = \frac{3\,W}{10^7\,I} = t$$

shows that t is the diagonal axis for both charts. Assuming L to 30 (L^3 to 27,000), W to 300,000, and I to 2400, the equation becomes

$$\frac{D'}{L'^3} = \frac{W'}{I'} = t \quad ,$$

where $$L'^3 = \frac{L^3}{5000}, \quad W' = \frac{W}{50{,}000}, \quad I' = \frac{I}{400}, \quad D' = \frac{16}{3} D \quad .$$

Fig. 14 is the chart, and a specimen solution is shown.

6. Concurrent Line Charts.

$$\frac{1}{f_1} + \frac{1}{f_2} = \frac{1}{f_3} \quad .$$

Equations of this sort occur in a surprising number of environments — electrical circuits, optics, mechanics, piping, etc. The basic form is

$$\begin{vmatrix} \frac{1}{f_1} & 0 & 1 \\ \frac{1}{f_2} & 1 & 0 \\ \frac{1}{f_3} & 1 & 1 \end{vmatrix} = 0 \quad ,$$

and the working form

$$\begin{vmatrix} f_1 & 0 & 1 \\ 0 & f_2 & 1 \\ f_3 & f_3 & 1 \end{vmatrix} = 0 \quad .$$

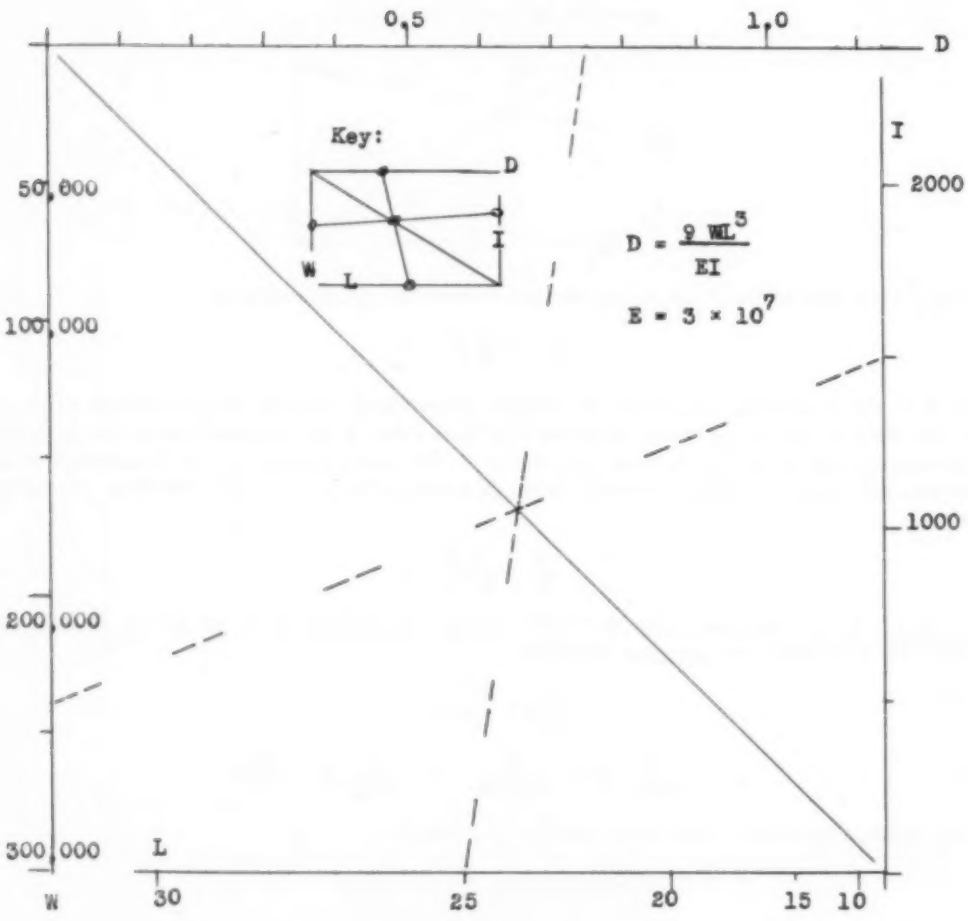


Figure 14

The line of zero abscissas carried f_1, the line of zero ordinates carries f_2, and the line along which ordinates and abscissas are equal — that is, the line bisecting the angle between the other axes — carries f_3. If the two first axes be set at 120^0 instead of 90^0, the units along the

three axes are equal, which greatly simplified graduation. A characteristic problem of the sort is

$$\frac{1}{x}+\frac{1}{y}=\frac{1}{z} \quad ,$$

the chart of which is Fig. 15. Extension to more than three variables is very much like that of earlier examples. For example, for $\frac{1}{x}+\frac{1}{y}+\frac{1}{z}+\frac{1}{u}=\frac{1}{v}$, we set up the sequence $\frac{1}{x}+\frac{1}{y}=\frac{1}{p}$, $\frac{1}{p}+\frac{1}{z}=\frac{1}{q}$, $\frac{1}{q}+\frac{1}{u}=\frac{1}{v}$, and get the chart shown in Fig. 16.

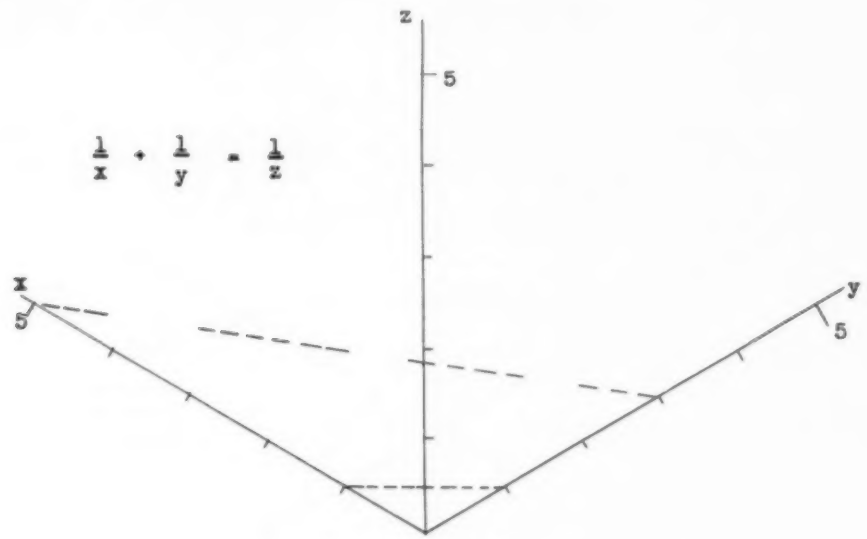


Figure 15

7. Curve Charts.

$$f_1(x)+f_2(y)\times f_3(z)=\phi(z) \quad ,$$

f_3 and ϕ being different functions of the same variable.

The basic form is

$$\begin{vmatrix} f_1 & 0 & 1 \\ f_2 & 1 & 0 \\ \phi & f_3 & 1 \end{vmatrix}=0 \quad ,$$

and the working form

$$\begin{vmatrix} f_1 & 0 & 1 \\ f_2 & 1 & 1 \\ \frac{\phi}{f_3+1} & \frac{f_3}{f_3+1} & 1 \end{vmatrix}=0 \quad .$$

The two parallel axes carry f_1 and f_2 respectively. The locus for f_3 is given parametrically by the two variable terms in the third row of the working form. Such a locus is generally a curve.

An illustrative example is

$$x^2+yz^2=z^3$$

with assumed ranges of x from 0 to 15, and y 0 to 20. Since X^2 then reaches 225, the equation is rewritten

$$40\left(\frac{x^2}{40}\right) + 4\left(\frac{y}{4}\right) z^2 = z^3$$

which leads to the working form

$$\begin{vmatrix} \frac{x^2}{40} & 0 & 1 \\ \frac{y}{4} & 1 & 1 \\ \frac{z^3}{4z^2+40} & \frac{4z^2}{4z^2+40} & 1 \end{vmatrix} = 0 \quad .$$

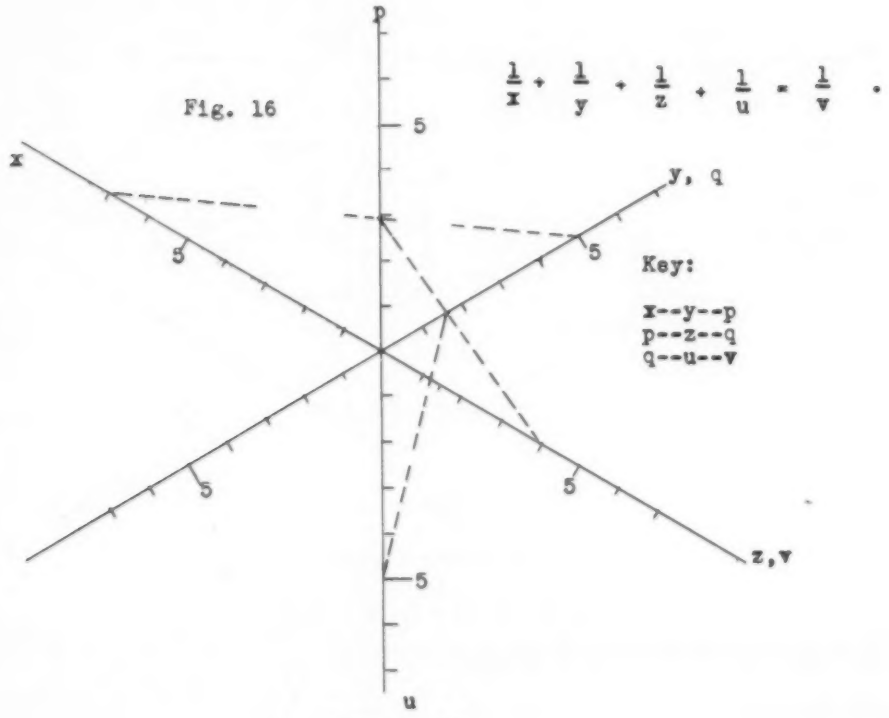


Fig. 16

$$\frac{1}{x} + \frac{1}{y} + \frac{1}{z} + \frac{1}{u} = \frac{1}{v} \quad .$$

Figure 16

The chart is Fig. 17. Points on the Z locus are easily found, in this case, without numerical computation by noting that for x = 0, then y = z, and that $x^2 = z^2 (z-y)$. If, then, x = 0 be joined to any point on the y- axis, the corresponding value of z (=y) lies somewhere on this line. Furthermore, if y = z-1, then x = z. Hence if y = z-1 be joined to x = z, the point z lies somewhere on this line. The intersection of two such lines gives the point for z. For example, to locate z = 10, first join x = 0 to y = 10; then join x = 10 to y = 9. The intersection of these two lines locates the point z = 10. Similar devices can be used to graduate the z locus throughout.

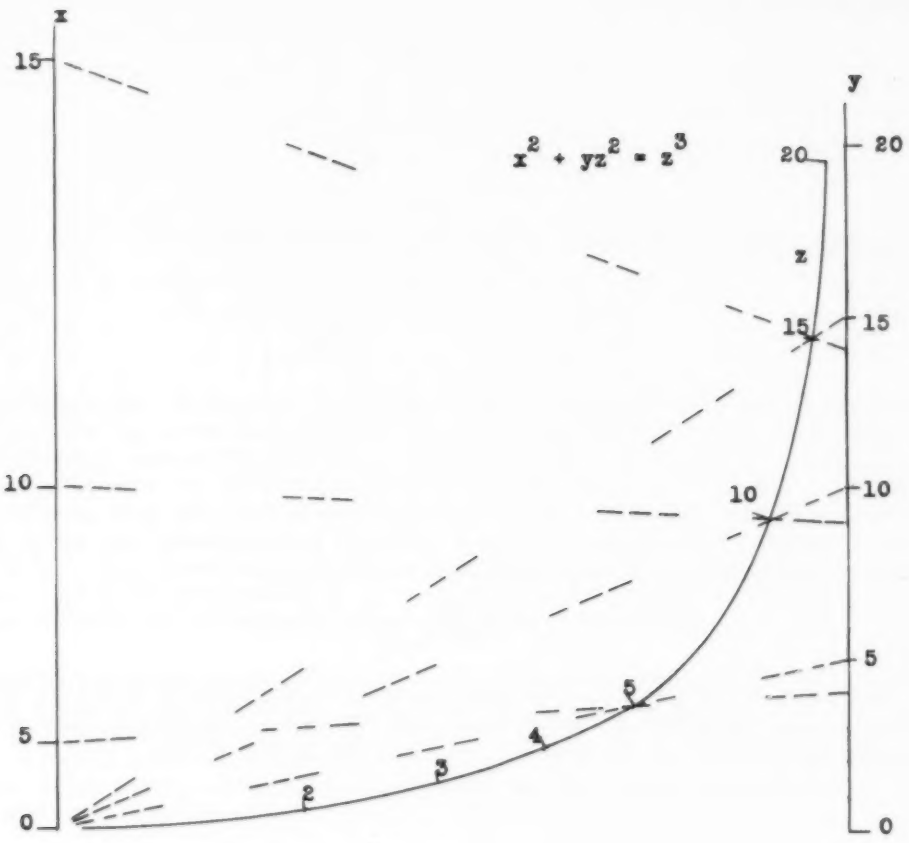


Figure 17

8. Combination Charts.

The various forms of primary charts shown can be combined, and are combined by the algebraic nature of the equations. For example, the equation

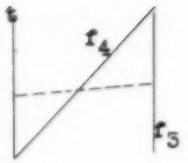
$$f_1(x) + f_2(y) = \frac{f_3(z)}{f_4(w)}$$

can be written as the sequence

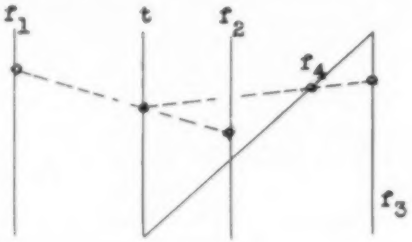
$$f_1(x) + f_2(y) = t \quad , \quad t = \frac{f_3(z)}{f_4(w)}$$

The first equation gives a parallel axis chart,

and the second gives an N-chart



Using the same t axis for both charts, we have the combination chart



Still another combination is illustrated by Bazin's Formula

$$v = \frac{87\sqrt{rs}}{0.552 + \frac{m}{\sqrt{r}}}$$

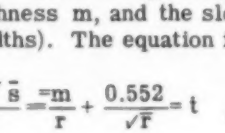
for the velocity, v, in feet per second, of flow through an open channel of hydraulic radius r, the banks having a coefficient of roughness m, and the slope of the channel being s ft. per thousand (s being measured in thousandths). The equation is first written

$$\frac{87\sqrt{s}}{v} = \frac{m}{r} + \frac{0.552}{\sqrt{r}} = t \quad ,$$

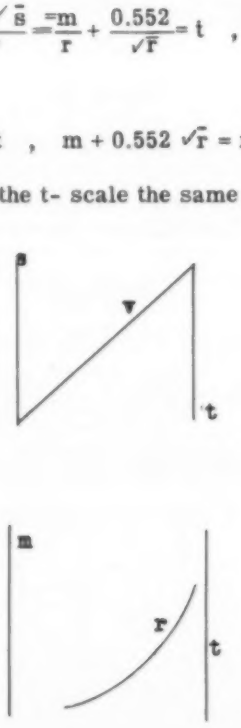
giving the two equations

$$87\sqrt{s} = vt \quad , \quad m + 0.552\sqrt{r} = rt \quad .$$

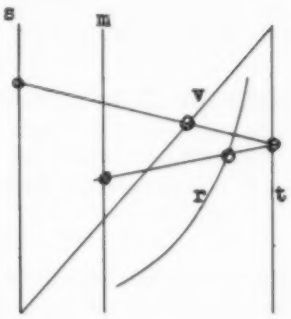
With proper choice of moduli to make the t- scale the same on both charts, the first equation gives the N- chart



and the second gives the curve chart



These are then combined into the single chart



This last chart, with the equation from which it arose, illustrates very well the power of the nomogram. Any one of the four variables can be evaluated, given values of the other three, with no algebraic manipulation whatever. It requires only a little attempt at such algebraic manipulation to show the force of this remark. Furthermore, the slide rule, or other computing machine, finds this type of calculation particularly confusing, because of the addition as well as the multiplications and divisions involved. Finally — and perhaps the greatest reason of all — it is almost impossible to misplace the decimal point in using the homogram, and every user of the slide rule or of the log table knows the danger of this kind of error — certainly every teacher is tragically aware of the ease of such an error.

It is evident that setting up the basic form, and the algebraic manipulations often necessary to set up the basic form, constitute the mainspring of the whole matter. Fortunately, there is no need to decide beforehand what type of graph is to be used — the determinant takes care of that. It is this feature of the determinant which leads the writer to prefer the determinant approach, as the reader will discover for himself if he will examine treatments of the subject by the similar triangle approach.

To use the nomogram requires no knowledge of nomography as a science — no knowledge other than that of reading a graduated scale is required. Since a chart can be reproduced inexpensively, it can be placed in as many hands simultaneously as desired. This may mean considerable saving in expense as compared with slide rules, computing machines, or even log tables — and may possibly mean the availability of less highly skilled computers.

Constructing nomographic charts in a practical environment is not economical unless the same formula is to be used for a large number of computations, or unless a considerable number of computers are to use the same formula simultaneously. But more and more design rooms, etc., do find that just such a need arises, and the nomogram is rapidly growing in usefulness.

One remark about a disadvantage of the nomogram. It is entirely non-pictorial — that is, it can not show relative variations, as does, say, the conventional temperature-time of day chart, or any chart of that sort. The nomographic chart is strictly a computing, rather than a picturing, device.

COMPONENTS OF DIGITAL COMPUTERS

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1. Digital Computers

As the name implies, digital computers calculate by manipulating the digits of quantities expressed in numerical form. Arithmetic operations such as addition and multiplication are performed in much the same way as in hand computations with pencil and paper. By way of contrast, in analog computers variables are represented by physical quantities such as length, angle or rotation, or voltage. These quantities then are constrained by appropriate connection of the computer components to follow the desired mathematical relations. Simple classic examples of digital and analog type computers are the desk calculator and the slide rule, respectively.

Computer designers have exercised considerable ingenuity in utilizing a great variety of physical devices and phenomena to mechanize the operations required of computers. In this paper are described in simple terms some of the components most commonly used at the present time in the modern high-speed digital machines.

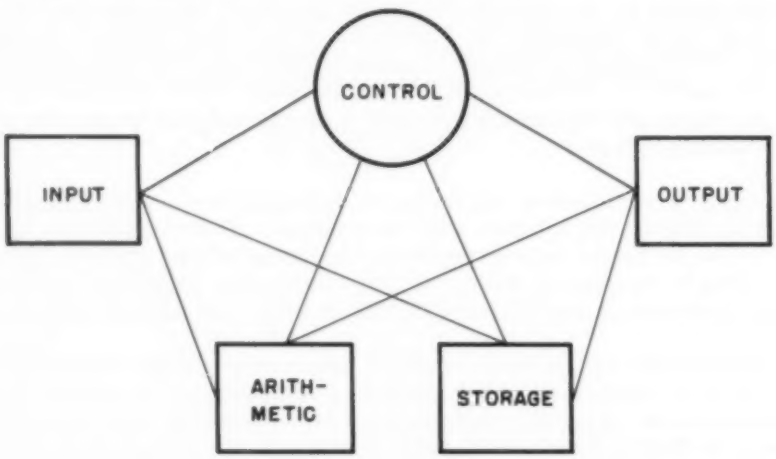


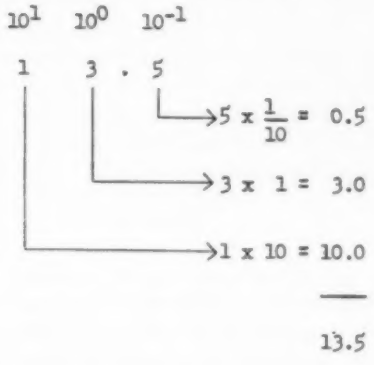
Figure 1. Principal functional elements of a digital computer

Figure 1 shows the principal functional elements of a digital computer. These are the input and output members, the arithmetic element, the storage element, and the control element. The input and output mechanisms are the means by which the computer communicates with the outside world. All numerical quantities associated with a problem, as well as detailed instructions as to how the machine is to make the calculation, are inserted through the input equipment. The arithmetic element performs the individual operations of addition, subtraction, multiplication, and division. The storage element, or memory, stores intermediate results as well as instructions and numerical data inserted initially. The output equipment then makes available in some suitable form the results of the calculation.

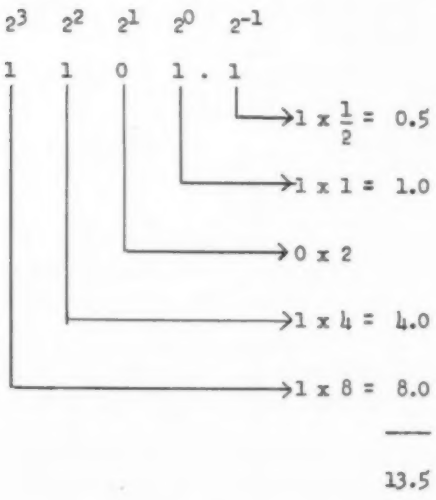
The control element is the brains behind the whole operation. It interprets instructions given the machine and supervises each step of the computation. Although the control element is not always a separate physical entity which can be isolated and labeled "control," some portions of the machine must be devoted to the control function.

2. Binary Arithmetic

Early in the design of a computer, two basic decisions must be made. One is the selection of the number system in which the machine calculates; the other is the choice between serial and parallel computation, the subject of the next section.



(a) Decimal representation of 13.5



(b) Binary representation of 13.5

Figure 2.

The majority of high-speed computers that have been built recently or are now under construction compute in the binary number system. In this system the digits of a number are coefficients of the various powers of two, rather than ten as in the decimal system. Also the digits can have only the values 0 and 1 in the binary system instead of the 0 through 9 of the decimal system. These properties are illustrated in Figure 2, which compares the decimal and binary representations of 13.5.

Perhaps the principal reason for the use of binary notation in computers is that only two distinct levels or states are required to represent a digit rather than the ten, for example, which would be required in the decimal system. Nearly all components suitable for use in high-speed computers have fundamentally only two stable states. Consequently they are well suited naturally for binary computation. A device having ten stable states would have to be built up from at least four bistable elements. Since four bistable elements are capable of representing $2^4 = 16$ separate levels — six more than the ten required — this constitutes somewhat inefficient use of equipment

Another important reason for computing in the binary system is the simplicity of binary arithmetic. For example, the binary multiplication and addition tables are simply as shown in Table 1.

$0 \times 0 = 0$	$0 + 0 = 0$
$0 \times 1 = 0$	$0 + 1 = 1$
$1 \times 1 = 1$	$1 + 1 = 10$
Binary multiplication table	Binary addition table

Table 1.

Some disadvantages also are associated with binary notation. Chief of these are the unfamiliarity of human operators with binary representation and the necessity for roughly three times as many digits as are required in the decimal system. Nevertheless, at the present state of the art the advantages of binary computation apparently outweigh the disadvantages sufficiently to recommend its use in high-speed general-purpose machines.

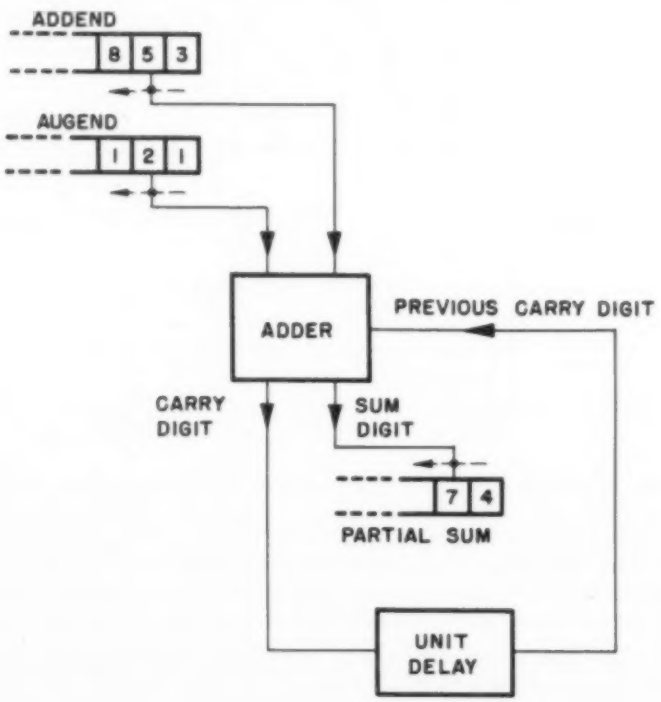


Figure 3. Schematic of a serial adder

3. Serial and Parallel Computation

The choice between serial and parallel computation concerns the manner in which the digits of numbers are handled. In serial operations, digits are processed sequentially; in parallel operations, simultaneously. Serial addition of two numbers, for instance, is performed by adding the digits of equal significance, one pair at a time. This operation is indicated schematically in Figure 3. A single-stage adder receives a pair of digits, one from each number being added, and produces a sum digit and a carry digit. The carry digit is delayed one time unit and transmitted to the adder along with the next pair of digits. Thus the sum is formed one digit at a time.

In parallel addition, one elementary adder is provided for each digit of the numbers being added. The schematic of a parallel adder is shown in Figure 4. To add numbers of n significant digits in this way requires n times as much equipment as serial addition. However, serial addition takes n elementary addition times, whereas parallel addition requires only one, plus sufficient time for all carries to propagate. Because of this advantage in speed, higher-speed machines in general make use of parallel arithmetic.

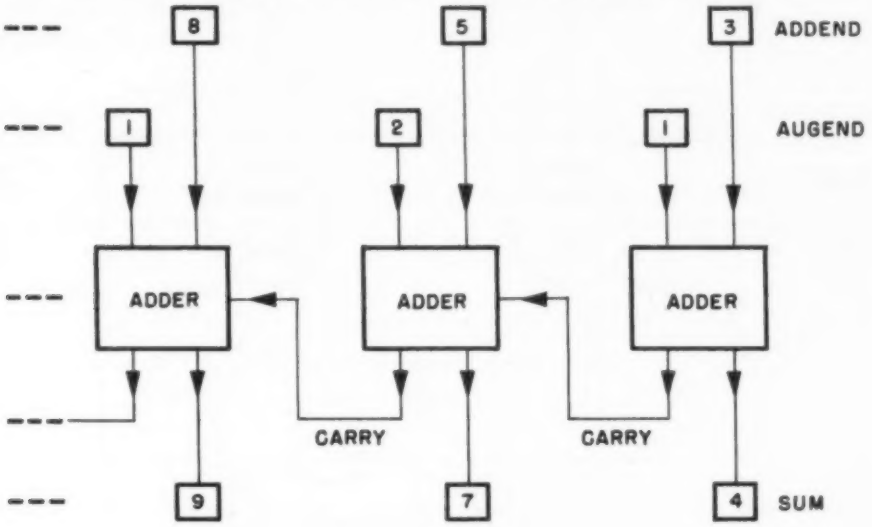


Figure 4. Schematic of a parallel adder.

4. The Flip-Flop Circuit

The workhorse of modern digital computers is the "flip-flop" circuit, a bistable circuit built around two vacuum tubes. A simplified diagram of one type of flip-flop is shown in Figure 5.

The tubes in a flip-flop operate in one of two conditions: conducting substantial current or negligible current. The amount of current which flows from plate to cathode in a vacuum tube is controlled by the voltage applied to its grid. If the grid potential is considerably negative with respect to that of the cathode, no current is permitted to flow, and the tube is said to be cut off. On the other hand, if the grid-to-cathode potential is slightly positive, the tube conducts current relatively well. The resistor values and supply voltages for the flip-flop circuit are so chosen that only two stable states are possible. In each of these, one tube is cut off and the other is fully conducting. In the schematic diagram, Figure 5, for instance, if tube 1 is conducting, current flow through its plate load resistor R_{L1} lowers the potential at the grid of tube 2 sufficiently to cut it off. Similarly if tube 2 is conducting, tube 1 is cut off.

The state of the flip-flop — that is, which of its tubes is conducting — may be changed by pulses

applied at the appropriate input. Suppose that tube 1 is conducting initially and tube 2 is cut off. A negative pulse applied at input 1 now will momentarily depress the grid of tube 1. This decreases the current passed by the tube, which raises the potential of its plate, and consequently raises also the potential at the grid of tube 2. Tube 2 then begins to conduct slightly. But this decreases the grid potential of tube 1 still further, again raising the potential at the grid of tube 2. This cumulative action, which actually requires an exceedingly small time, continues until tube 2 is fully conducting and tube 1 is cut off. Thus the state of the flip-flop has been changed from tube 1 conducting to tube 1 cut off by a negative pulse applied at input 1. Similarly now a negative pulse at input 2 will revert the flip-flop to its initial state with tube 1 conducting.

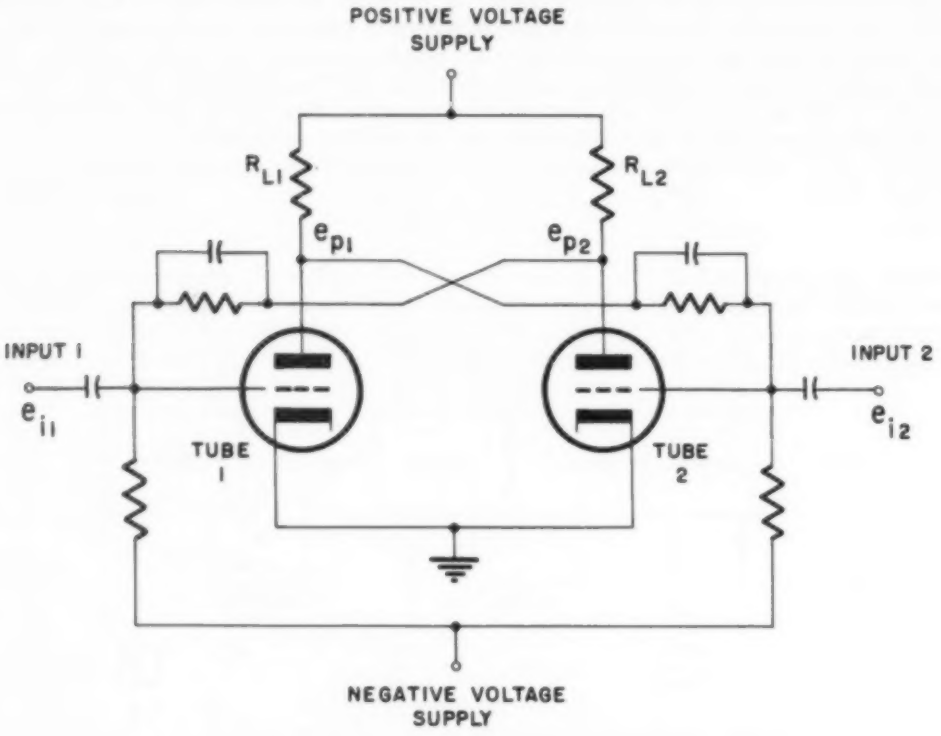


Figure 5. Schematic of a flip-flop circuit

5. Counters

If input terminals 1 and 2 are connected together, every negative pulse at the combined input will change the state of the flip-flop, regardless of its initial condition. Suppose, for instance, that tube 1 is cut off initially so that e_{p1} is near the positive supply voltage. Plate 1 then is said to be "high", and this condition is taken to represent a ONE. The potential at plate 2, however, is "low" since tube 2 is conducting. This low voltage corresponds to a ZERO. A pulse applied to the combined input, now, changes the state of the flip-flop. Thus e_{p1} becomes ZERO and e_{p2} ONE. The next pulse returns the circuit to its original state with e_{p1} ONE and e_{p2} ZERO. This mode of operation is summarized in Table 2.

The drop in potential of plate 2 at pulses 2, 4, 6, ... may itself be shaped into a negative pulse suitable for triggering another flip-flop. Figure 6 shows a chain of three flip-flops, the last two of which receive as inputs pulses from the second plate of the succeeding stage. Pulses applied to the input of the first flip-flop affect the circuit as indicated in Table 3. Here all

Pulse Number	Plate 1, e_{p1}	Plate 2, e_{p2}
0	1	0
1	0	1
2	1	0
3	0	1
4	1	0
⋮	⋮	⋮

Table 2

even-numbered plates are taken to be ZERO initially. The flip-flops evidently form a three-stage binary counter capable of counting from 0 through 7. The condition 011 at pulse 3, for instance, is the binary representation of 3. The addition of more flip-flops, of course, would enable the circuit to count higher. Such combinations of flip-flops are capable of counting at rates as high as 10 million pulses per second — 10,000 times faster than the electromechanical relays used for similar purposes in earlier computers.

Pulse Number	e_{p6}	e_{p4}	e_{p2}
0	0	0	0
1	0	0	1
2	0	1	0
3	0	1	1
4	1	0	0
5	1	0	1
6	1	1	0
7	1	1	1
8	0	0	0
9	0	0	1
⋮	⋮	⋮	⋮

Table 3

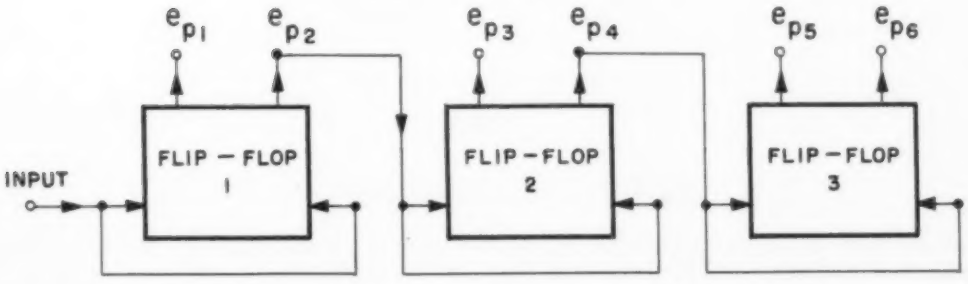


Figure 6. A three-stage binary counter

6. Switches and Gates

Switches and gates constitute another important class of computing circuits. Their principal functions are (1) control of the flow of information within a computer and (2) conversion of data from one form to another. Two main types of switching circuits are in common use at the present time in high-speed computers.

One type utilizes multi-element vacuum tubes in various combinations to mechanize the required operations. Consider as an example the circuit shown in Figure 7. The output e_0 of this circuit depends on the two inputs, e_1 and e_2. If the potentials at both inputs are low (considerably negative with respect to the common cathode potential) the tubes are cut off and e_0 is high (equal to the positive supply voltage). However, if either or both of the inputs are high, at least one tube conducts. Current flow through the plate resistor R_L then drops e_0 to a relatively low potential. Again let the high condition represent ONE and the low condition ZERO. With this convention, the operation of the circuit can be summarized as in Table 4.

e_1	e_2	e_0
0	0	1
1	0	0
0	1	0
1	1	0

Table 4.

In similar fashion, logical functions of any number of input quantities can be formed by various combinations of triodes. The circuits often may be simplified, however, by the use of pentodes or other special-purpose tubes whose behavior depends on two or more control grids.

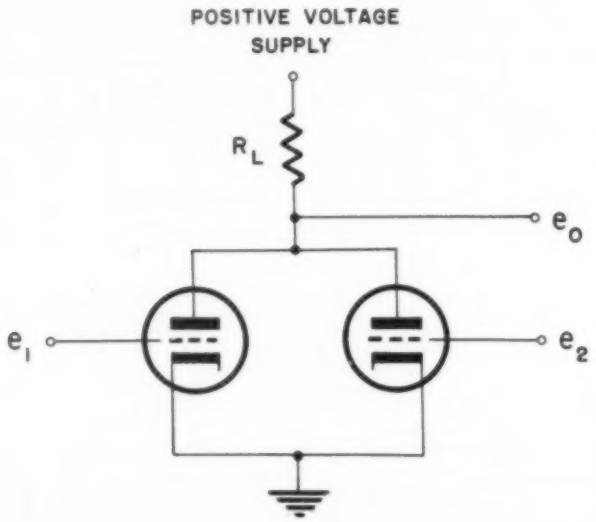


Figure 7. A vacuum-tube gate circuit.

In another class of switching circuits logical functions are mechanized by means of networks composed of resistors and rectifiers. A rectifier is essentially a nonlinear resistor which conducts current effectively in only one direction. Rectifiers suitable for high-speed switching are vacuum-tube diodes and semi-conductor rectifiers of either the point-contact (germanium, silicon) or barrier-layer (selenium) type. Any of these may be represented schematically as indicated in Figure 8a. In this paper the direction of easy current flow is taken to be with the arrow, although the opposite convention sometimes is used also.

Figure 8b shows a circuit using two rectifiers to produce an output which is ONE if and only if both inputs are ONE. Suppose first that e_1 and e_2 are both 0 (corresponding to ZERO's). Since the top end of the resistor R is maintained at a positive 30-volt potential, the arrow

ends of the rectifiers tend to be positive with respect to the other ends. Current then flows easily through the rectifiers in the arrow direction. The resistance R is sufficiently large with respect to the resistance of the rectifiers when conducting in this direction that very little voltage appears across the rectifiers. The output voltage e_0, consequently, is practically the same as e_1 and e_2 which are 0.

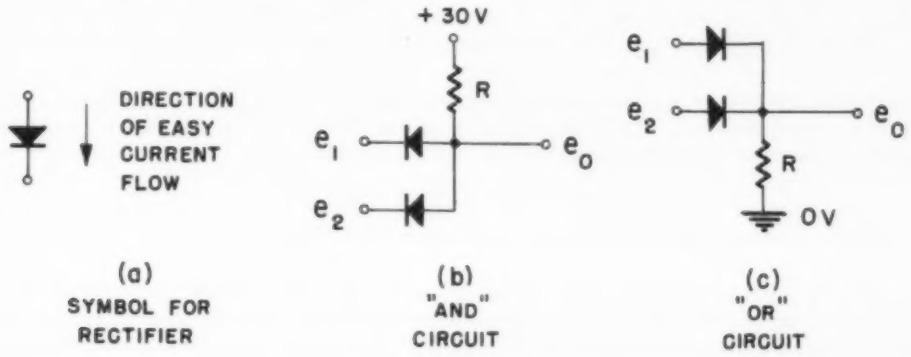


Figure 8. Rectifier symbol and logical circuits.

Now let e_1 be raised to +30 volts, corresponding to a ONE, with e_2 held at 0. The rectifier connected to e_2 still behaves essentially as a short circuit so that e_0 is nearly 0. Thus the arrow end of the rectifier connected to e_1 is negative with respect to e_1. This rectifier then acts as an open circuit and disconnects e_1 from the circuit. A similar situation, in which e_0 is again 0, results if e_1 is 0 and e_2 is +30. On the other hand, if both e_1 and e_2 are +30 volts, the entire circuit including e_0 is at this potential. In this case then, e_0 represents a ONE. Table 5 lists the four possible input combinations and the result of each.

e_1	e_2	e_0
0	0	0
1	0	0
0	1	0
1	1	1

Table 5.

The simple equation

$$e_0 = e_1 e_2$$

describes the operation of this circuit when the e's are considered to be either 0 or 1. The circuit sometimes is called a logical AND circuit because its output is ONE only if e_1 and e_2 are ONE. The number of terms in the product, or AND relationship, can be increased by connecting more inputs to e_0, each through a separate rectifier. With n such inputs, the output of the circuit is

$$e_0 = e_1 e_2 e_3 \ldots e_n.$$

Another basic rectifier circuit is the logical OR circuit. A two-input OR circuit is shown in Figure 8c. Its operation is summarized in Table 6, as can be easily verified by considering each input combination separately.

The circuit can be described mathematically by the equation

$$e_0 = e_1 \vee e_2,$$

e_1	e_2	e_0
0	0	0
1	0	1
0	1	1
1	1	1

Table 6.

in which "v" by definition means that the combination of terms it links is ONE if either or both of the adjacent terms are ONE. Otherwise the combination is ZERO. The circuit of Figure 8c is easily extended to mechanize the relationship

$$e_0 = e_1 \text{ v } e_2 \text{ v } e_3 \ldots \text{ve}_n$$

by the addition of more inputs with their associated rectifiers.

Mathematical descriptions of the sort given above for the AND and OR circuits are of considerable value in the design of circuits to perform specified logical operations. A branch of mathematics known as logical algebra or symbolic logic deals with just such relationships as these. Complicated operations may be described precisely and compactly in terms of the symbols of logical algebra. These descriptions then may be simplified and manipulated according to well defined rules to place them in forms suitable for realization by circuits of the type available.

7. Storage Systems

Performance requirements placed on the storage facilities of a computer often cover such extreme ranges that no single storage medium is satisfactory. For this reason two, and sometimes three, distinct types of storage usually are provided. For example, some sort of high-speed storage, which permits quick insertion and withdrawal of information, is needed to hold intermediate results and current instructions. This device need not have large capacity, but the stored information must be readily accessible so as not to slow down calculation excessively. On the other hand, if the computer must use extensive mathematical tables or handle large quantities of data in the solution of a problem, the amount of information which must be stored may be very large. In this case high capacity is necessary, but speed requirements usually are not stringent.

Typical low-speed stores hold information as holes punched in paper tape or as magnetized areas on tape or wire. Devices utilizing these principles are capable of storing great quantities of information compactly and cheaply. However, the time required to locate and read some particular piece of information may be relatively long. The majority of high-speed storage systems of the present time are built around rotating magnetic drums, acoustic delay lines, or electrostatic storage tubes. These devices are described in greater detail in the remainder of this paper.

Two important characteristics of a high-speed storage system are its access time and its capacity. Access time is the time required to withdraw a number from storage and make it available for use in some other portion of the computer. Since this operation usually occurs at least once for every basic operation in a calculation, the access time is an important factor in determining the calculating rate of a machine.

The capacity of a storage system is measured in terms of the number of binary digits it can store. The high-speed stores of modern computers hold, as a rough average, about one thousand numbers or words. If the numbers consist of 40 binary places, the store then has a capacity of about 40,000 binary digits, or bits. It has been estimated that the average printed page contains an amount of information equivalent to 10 or 20 thousand bits. A typical

computer thus has rapid-access storage for an amount of information roughly equal to from 2 to 4 printed pages.

Storage systems may be classified as cyclic or noncyclic. In a cyclic system stored information circulates continuously, passing reading and writing stations periodically. The most common cyclic systems utilize rotating magnetic drums and acoustic delay lines. Examples of noncyclic systems are those employing flip-flop registers and electrostatic tubes. The access time of a cyclic system depends upon the location of a particular piece of information when called for and upon the circulation rate. If information is placed in storage without regard to the sequence in which it is to be used, the average access time is just half the period of circulation. Some improvement can be obtained, however, by careful coding and the use of multiple reading stations. The access time of a noncyclic system is essentially independent of the storage location. For this reason noncyclic systems sometimes are referred to as random access stores.

8. Magnetic-Drum Storage

The heart of the magnetic-drum storage system is a rotating cylindrical drum whose surface is coated with a magnetizable material. Information is placed in storage on the drum by a recording or writing head which magnetizes a small area of the surface, usually to saturation in one direction or the other. A reproducing or reading head placed almost a full revolution away from the writing head produces from these magnetized areas signals corresponding to the stored information. An erasing head may be provided between reading and writing heads to demagnetize the surface or to place it uniformly in some preassigned condition. The necessary equipment for one track is indicated schematically in Figure 9. The auxiliary circuitry on the right provides for re-recording information as read when it is to be re-stored without modification.

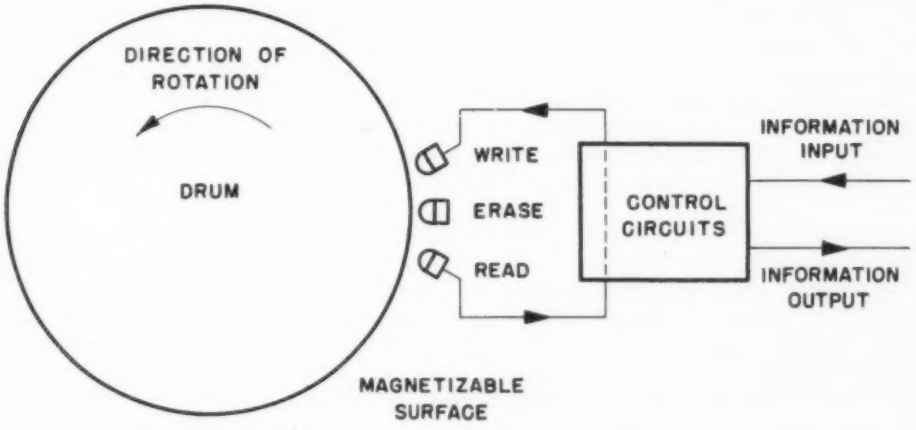


Figure 9. Schematic of a magnetic-drum storage system.

A reasonable linear velocity for the drum surface is about 1600 inches per second when operating at a density of 80 binary digits per inch. Thus the basic information handling rate is 128,000 bits per second. A drum able to store 1024 bits in one track would have a circulation period of $\frac{1024}{128{,}000}$ or 8000 microseconds. The average access time then is 4000 microseconds. The drum would be just over 4 inches in diameter to hold the 1024 bits. Tracks of digits can be stacked along the drum at the rate of about 10 per inch. A drum which holds 1024 40-digit words need be only about 4 inches long and 4 inches in diameter.

9. Acoustic Delay Line Storage

The acoustic delay-line store is a cyclic system similar to the magnetic-drum store. In this system, however, information is stored as mechanical disturbances which travel from one end of a stationary storage medium to the other. In most practical realizations, the delay line consists of a tank of mercury with quartz-crystal transducers at both ends. The crystal at the sending end converts electrical pulses to pulses of acoustic energy. The pulses propagate down the column of mercury at the velocity of sound to the receiving crystal, which converts the pulses back to electrical signals. These signals then are fed back to the sending end to form a closed circuit. As in the case of the magnetic-drum store, apparatus is included to modify the stored information before re-insertion into the delay line or to read it out for use in the computation.

The velocity of propagation of sound in mercury is about 57,000 inches per second. Pulse densities of 70 per inch are usable so that rates of 4 million bits per second can be achieved. A line which could hold 1024 bits would have a circulation period of 256 microseconds, or an average access time of 128 microseconds. It would be about 15 inches ling.

Practical difficulties with mercury delay-line storage are caused by the fact that the velocity of sound in mercury varies with temperature. As a result, the effective length of the line and the number of pulses it holds depends on the temperature. Lines cannot be built to store more than a few thousand pulses without extremely accurate temperature controls. Magnetic drums, on the other hand, have been built which hold several million bits of information.

10. Electrostatic Storage

In a third common storage system, information is stored as electrostatically charged spots on the screen of a cathode-ray tube. Most such tubes contain an electron gun at one end which produces a thin stream of electrons aimed in the general direction of the storage surface at the other end. Deflection plates are placed between the gun and the storage surface so that the beam can be directed to any desired region of the surface. Information is stored typically in a square array of perhpas 1024 discrete spots. The surface may be simply a dielectric sheet or a mosaic of tiny insulated metallic beads. Read out of the information is accomplished by means of a metallic plate, placed against the back of the surface, which couples capacitively with the individual storage elements.

Several different schemes, utilizing either standard cathode-ray tubes or specially designed tubes, have been developed for electrostatic stores. These systems differ considerably in the details of writing, reading, and regenerating information on the storage surface. However, most depend on rather complicated secondary-emission phenomena for the establishment of two stable charged states at a given spot on the storage surface. Present systems require considerable auxiliary electronic equipment, and consequently they are comparatively expensive and less reliable than magnetic-drum stores, for example. However, because they make possible very short access times — of the order of 25 microseconds — electrostatic stores are incorporated in many of the faster machines that have been built recently or are now under construction.

11. Future Possibilities

The components and technics described in the preceding sections of this paper are those which have already found extensive use in computers. In closing, mention should be made of some of the devices which seem to hold considerable promise for the future. The transistor, for example, a product of recent solid-state physics research, can perform most of the tasks now assigned to vacuum tubes in computers.

Present transistors consist of a small piece of germanium to which are attached three electrodes. As in the vacuum-tube triode, one electrode controls the electrical characteristic

between the other two. However, the transistor, in contrast to its vacuum-tube counterpart, does not require a heated cathode with the associated power loss and possibility of failure. Also, the transistor is much simpler mechanically than the vacuum tube since it contains no complicated and delicate grid structure. Accordingly, the transistor is expected to make possible computing circuits with greatly increased reliability, that occupy only a fraction of the space now required by equivalent vacuum-tube circuits.

Considerable effort is currently being directed toward utilizing the hysteretic characteristic of ferromagnetic materials for computer applications. Hysteresis effectively gives the material the ability to "remember" its recent history. Consequently it is capable of storing information. Stores built on this principle consist of tiny toroidal cores of magnetic material arranged in a grid in, say, the x-y plane. Each core stores one binary digit which may be written or read on specification of its x and y coordinates. A three-dimensional array of the cores, built up from the two-dimensional grids, appears to provide a practicable, compact means for random-access parallel storage. Other devices which have been successfully built from magnetic cores include switches and bistable circuits analogous to flip-flops.

Some insulating materials exhibit dielectric hysteresis similar to the magnetic hysteresis of ferromagnetic materials. These materials, of which barium titanate is an example, are called ferroelectric. Storage systems and switches have been built from such materials, which are exactly analogous to the magnetic circuits mentioned in the preceding paragraph. For example, in one recently reported device, storage for 256 bits is provided on a single crystal of barium titanate 0.45 inch square and 0.01 inch thick. Sixteen straight conductors are placed on one side of the crystal and 16 on the other side perpendicular to the first set. Small capacitors, each of which stores one bit, thus are formed at the 256 intersections of the conductors. Although much work remains to be done in the development of suitable ferroelectric materials, results achieved so far are very encouraging.

MATRIX ITERATION METHOD FOR CALCULATING CRITICAL FREQUENCIES OF VIBRATORY SYSTEMS IN BENDING

GEORGE SONNEMANN*

Introduction:

As a rotary system is set in motion, one can observe a range of speeds at which the vibratory forces become large. The speed at which the amplitude of vibration becomes large is called the critical speed. We must remember here that definite distinction exists between the actual physical problem and the idealized mathematical problem. The difference between the two forms, that is, the difference between the mathematical model and the physical problem, are the approximations and assumptions that are made to mold the physical problem into one that can be dealt with mathematically.

For the remainder of our discussion here let us make two major assumptions, namely, that the deflections are small and that the damping can be neglected. The first of these assumptions will make it impossible to determine the amplitude of vibration whereas the second assumption will permit the amplitude to increase without limit at the critical speed. The omission of damping permits us to simplify the mathematics but does not alter the general method of solution. The restriction to small amplitudes means that only linear terms enter into the equations. If non-linear terms were considered, then the mathematical problem would become much more complex.

Fundamental Theory of Vibration. Problem

Let us look at the vibration of beams here. We recall that a beam is a continuous elastic medium and thus is capable of assuming an infinite number of deflection curves (modal shapes). This we subsequently will see is not desirable for our work. Hence let us consider that the physical beam is replaced by a beam consisting of a number of segments which possess rigidity but no mass and a number of masses act upon this beam. Or we could say: Let the mass of the beam be included into the applied loads or lumped into a number of concentrated masses and then divide the massless beam into a finite number of segments, these segments possessing a flexural rigidity.

Suppose then the beam vibrates laterally. We recall that the internal resisting moment at some point x_i with the deflection y_i is

$$M_i\Big]_{INT.} = EI\,\frac{\partial^2 y_i}{\partial x_i^2} \tag{1}$$

Let the externally applied moment be M, and then for equilibrium to exist the external moment must balance the internal moment. Thus we get

$$M_i\Big]_{EXT.} = EI\,\frac{\partial^2 y_i}{\partial x_i^2} \tag{2}$$

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Let us assume for the sake of simplicity here that the moment of inertia I is a constant along the length of the beam. If we differentiate eq. (2) twice, we obtain

$$EI \frac{\partial^4 y_i}{\partial x_i^4} = \frac{\partial Y}{\partial x} = \sum_{n=1}^{i} w_n$$

where $\sum_{n=1}^{i} w_n$ is the loading of the beam from the origin to the point x_i. Since we are dealing with a dynamic system the forces that are acting are given by Newton's Law and can be represented by using D'Alembert's principle, namely

$$w_i = m_i a_i = m_i \frac{\partial^2 y_i}{\partial t^2} \tag{3}$$

where m_i is the mass at the point $x = x_i$. Thus the differential equation of motion becomes

$$EI \frac{\partial^4 y_i}{\partial x_i^4} = \sum_{n=1}^{i} m_n \frac{\partial^2 y_n}{\partial t^2} \tag{4}$$

The solution to the above equations can be found in several textbooks dealing with vibration problems. What we would like to do here however is to pursue a different avenue of approach. To do this most effectively here let us invoke the use of an example. In order to have one that we can cope with most readily let us choose a cantilever beam. In order not to have too many points of loading, let us restrict the loading to just two mass points. Suppose now that we have the deflected beam as shown in Fig. 1.

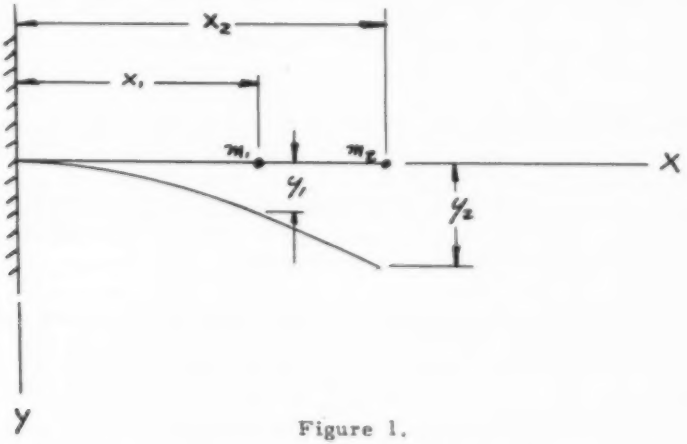

Figure 1.

Recall here the notion of influence coefficients. The influence coefficient is defined as the (static) deflection at a point due to a unit load at the same or some other point. In other words, this means we apply unit loads and observe the deflections caused by these. One advantage of this is that when the loads are different than unity we need only multiply the deflections by the magnitude of the loads that are actually applied. Thus the deflections at point 1; i.e., $x = x_1$, of the beam shown in Fig. 1 may be written

$$y_1 = a_{11} P_1 + a_{12} P_2 \tag{5}$$

where a_{11} is the deflection at point 1 due to the unit load at point 1 and a_{12} is the deflection at point 1 due to the unit load at point 2 and P_1 and P_2 are the loads at points 1 and 2 respectively. By multiplying the deflections caused by the unit loads by the actual loads, the deflections due to the actual loads are found and then by the principle of superposition the sum total deflection is obtained. In our dynamic problem the load P_1 in which we are interested is equal to the mass times the maximum acceleration at point 1 and if we assume the beam vibrates harmonically we may write

$$P_1 = m_1 a_1 = m_1 y_1 \omega^2 \tag{6}$$

and similarly

$$P_2 = m_2 a_2 = m_2 y_2 \omega^2 \tag{7}$$

Combining eqs. (6) and (7) and putting them into eq. (5), we obtain

$$y_1 = m_1 a_{11} y_1 \omega^2 + m_2 a_{12} y_2 \omega^2 \tag{8}$$

For the deflection at point 2 we obtain a similar equation, namely,

$$y_2 = m_1 a_{21} y_1 \omega^2 + m_2 a_{22} y_2 \omega^2 \tag{9}$$

Let us generalize for a moment. Suppose we were to consider a beam with very many points of mass loading, then we would obtain a whole set of equations for the deflections of the type

$$\begin{aligned}
y_1 &= a_{11} m_1 y_1 \omega^2 + a_{12} m_2 y_2 \omega^2 + \cdots + a_{1n} m_n y_n \omega^2 \\
y_2 &= a_{21} m_1 y_1 \omega^2 + a_{22} m_2 y_2 \omega^2 + \cdots + a_{2n} m_n y_n \omega^2 \\
&\vdots \qquad\quad \vdots \qquad\quad \vdots \qquad\quad \vdots \\
y_n &= a_{n1} m_1 y_1 \omega^2 + a_{n2} m_2 y_2 \omega^2 + \cdots + a_{nn} m_n y_n \omega^2
\end{aligned} \tag{10}$$

Notice that the above form a set of homogeneous algebraic equations with each equation containing the frequency ω. Also note that in each equation one of the deflections appears on both sides. Let us write the above equation in matrix form.

A matrix looks somewhat like a determinant, the difference being that one can manipulate or perform mathematical operations with a matrix which one can not do with a determinant. Thus the above system of equations may be written, after dividing through by ω^2, in the notation of matrices as*

$$\frac{1}{\omega^2} \begin{pmatrix} y_1 \\ y_2 \\ \vdots \\ y_n \end{pmatrix} = \begin{pmatrix} a_{11} m_1 y_1 + a_{12} m_2 y_2 + \cdots + a_{1n} m_n y_n \\ a_{21} m_1 y_1 + a_{22} m_2 y_2 + \cdots + a_{2n} m_n y_n \\ \\ a_{n1} m_1 y_1 + a_{n2} m_2 y_2 + \cdots + a_{nn} m_n y_n \end{pmatrix} \tag{11}$$

*This step can be made plausible by abstract notions of algebra. In particular if one were to

Let us next define matrix multiplication. If we look at the right side of eq. (11), we notice that every line or row has terms multiplied by y_1, y_2, y_3 to y_n in succession. We now want to be able to signify this multiplication. We define the multiplication as shown by eq. (12):

$$\frac{1}{\omega^2}\begin{pmatrix} y_1 \\ y_2 \\ \vdots \\ y_n \end{pmatrix} = \begin{pmatrix} a_{11}m_1 & a_{12}m_2 & \cdots & a_{1n}m_n \\ a_{21}m_1 & a_{22}m_2 & \cdots & a_{2n}m_n \\ \vdots & & & \\ a_{n1}m_1 & a_{n2}m_2 & \cdots & a_{nn}m_n \end{pmatrix}\begin{pmatrix} y_1 \\ y_2 \\ \vdots \\ y_n \end{pmatrix}. \qquad (12)$$

or in symbolic form

$$\frac{1}{\omega^2}(A) = (B)(A) \qquad (13)$$

where we shall consider the matrix (B) as an operator.* Now the multiplication is defined as we see by comparing eq. (11) and (12). The matrix of the deflections at the right side is a column matrix, that is, it consists of only one column. The column matrix multiplies each row of the other matrix, the first term in the column the first term in the row, the second term in the column the second term in the row and so forth. Or we bodily pick up the column and move it over the other matrix after turning it and then let the multiplication proceed down the columns. We thus have a multiplication scheme.

Now how does all this help in our problem here? Let us oonsider eq. (13). Here we see in symbolic form that we have a matrix (A) operated upon by matrix (B) set equal to matrix (A) again times a scalar, in this problem $1/w^2$. This type of problem is known as a characteristic value problem. The solution of the problem is as follows: Matrix (B) operates on an assumed matrix (A), then a scalar is extracted so that we get back matrix (A). For the vibratory system, due to the assumption of small deflections, we cannot get the absolute values of the deflections but only the relative values and generally one gets the so-called normalized coordinates. By this we mean, that to get relative normalized coordinates one deflection is assumed to be unity, usually at one end of the beam. Thus the magnitude of the scalar is so chosen as to make the one deflection unity. It becomes evident that we have no guarantee that by assuming matrix (A) and operating upon it by (B) and then extracting the scalar that we must get back the matrix (A). In fact we usually do not since we make the incorrect assumption for the matrix (A). However, it is possible to show that if we iterate, the solution will converge to the fundamental mode. In other words we assume values to make up matrix (A). Operate upon it, extract the $1/w^2$. Then we use this new matrix (A) and repeat the procedure until the matrix (A) before and after being operated upon is the same.

Illustrative Problem and Extension of the Theory

Let us demonstrate the above argument by means of a very simple example. Consider a cantilever beam as shown in Fig. 2.

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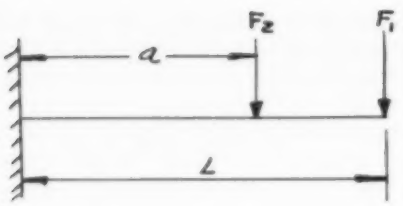
consider a system of homogeneous equations to form a set. However, for our purposes here we may just follow the notion of matrices as given by Aitken[2] in which the matrix (B) of eq. (13) is considered as an operator and the whole notation is a convenience to save writing. For the purposes of our discussion this will suffice.

*Recall here an operator is a quantity that when operating on a function gives us back some other function. We recall the differential operator D as an example.



E = 30,000,000 psi.
F_1 = 100 lbs.
F_2 = 200 lbs.
L = 10 in.
a = 6 in.
I = 1 in^4.

Figure 2.

The dimensions and physical constants are shown in Fig. 2. If we calculate the influence coefficients, the equations for the deflections become:

$$y_1 = 11.11(10^{-6})P_1 + 4.8(10^{-6})P_2$$
$$y_2 = 4.8(10^{-6})P_1 + 2.4(10^{-6})P_2 \qquad (14)$$
$$P_1 = \frac{F_1}{g}\omega^2 y_1 \; ; \quad P_2 = \frac{F_2}{g}\omega^2 y_2$$

Thus we get

$$y_1 = \left[11.11\,y_1 + 9.6\,y_2\right] 10^{-6}\frac{\omega^2}{g}$$
$$y_2 = \left[4.8\,y_1 + 4.8\,y_2\right] 10^{-6}\frac{\omega^2}{g} \qquad (15)$$

or

$$\lambda^2 y_1 = 11.11\,y_1 + 9.6\,y_2$$
$$\lambda^2 y_2 = 4.8\,y_1 + 4.8\,y_2 \qquad (16)$$

where

$$\lambda^2 = \frac{10^6 g}{\omega^2}$$

Writing this in matrix form, we get

$$\lambda^2 \begin{pmatrix} y_1 \\ y_2 \end{pmatrix} = \begin{pmatrix} 11.1 & 9.6 \\ 4.8 & 4.8 \end{pmatrix} \begin{pmatrix} y_1 \\ y_2 \end{pmatrix} \qquad (17)$$

We now assume a reasonable value for the deflections. It is clear that the deflection at the free end is the largest so that we need only make sure that our assumption at least agrees with that. As to the relative values, we may choose them arbitrarily. The closer the guess the quicker the end result is obtained but that is all. For this problem we chose one value to be unity and the other value to be five, or $y_1 = 5$; $y_2 = 1$.

With the assumed value of $y_1 = 5$ and $y_2 = 1$ we see that the multiplication gives the following: (11.11) times (5) is 55.55 plus (9.6) times (1) equals 65.15. In the second row (4.8) times (5) is 24 plus (4.8) times (1) gives 28.8.

The iteration is put into a table to conserve space. All the work is carried out on a calculating machine and only the work recorded here is the work that needs to be copied down. For this particular problem a Marchant calculating machine was used but a Friden will also serve the purpose. It may be pointed out here that the non-automatic machine will do, in other words the cheapest type of machine presently on the market will suffice. We now normalize column (2) by dividing through by 28.8 and we obtain column (3). We observe that column (3) and (1) do not agree. Hence, we use the new values for y_1 and y_2 obtained in column (3) and repeat the operation. Column (4) is the result which is normalized in column (5). We see we obtain much closer agreement. Repeating the cycle once more tells us we have converged to our answer. Notice that the amplitude ratios did not change a great deal after the first iterative step but the frequencies did. If we now solve for the critical speed, we obtain:

$$\lambda^2 = \frac{10^4 g}{\omega^2} = 15.446$$

$$\omega^2 = 647 g$$

$$\omega = 4772 \text{ C.P.M.}$$

The answer obtained by other methods is [3]:

Raleigh:	4775 cpm
Stodola:	4772 cpm
Prohl:	4775 cpm

We see that the answer can be obtained quite rapidly and be carried out by a person not at all familiar with the technical aspects of the problem. However, the method would have limited advantage if one could only obtain the first mode frequency. The Rayleigh method does not

TABLE I

	1	2	3	4	5	6	7
	(A)	(λ^2A)	(A)	(λ^2A)	(A)	(λ^2A)	(A)
y_1	5	65.15	2.26	34.20	2.218	34.24	2.217
y_2	1	28.8	1.00	15.64	1.00	15.446	1.00
λ^2			28.8		15.64		15.446

permit an evaluation of the higher modes. The Stodola method does permit evaluation of higher modes in a modified form. The Prohl method lends itself quite well to finding higher modes except that a rather tedious guessing game must be resorted to. The matrix method can be used easily to find the higher modes. Here a number of different methods can be used to permit the evaluation of the higher modes. The choice of the method depends upon the problem, the person applying the method and the closeness of the roots.

Let us explore one method here. The theory behind the method will not be explained in detail here. Essentially the basis of the subsequent discussion rests upon the theory of normalized coordinates and their orthogonality. We might recall here that for any vibratory system it is

possible to obtain a set of coordinates that are the so-called normal coordinates. If these coordinates are used then and if the system vibrates at the second natural frequency, for example, no component of the other frequencies will exist. This is the orthogonality condition. We now rewrite the matrix equation in a different form so that we obtain a solution in a modified form. We recall that we had the equation

$$\frac{1}{\omega^2}(A) = (B)(A) \tag{13}$$

The matrix (A) is a column matrix sometimes referred to as a modal column since the deflections of the modes are found herein. One can show that the critical frequency or characteristic value w^2 can be obtained by changing the order and form of the multiplication into the one below, namely,

$$\frac{1}{\omega^2}(E) = (E)(B) \tag{18}$$

where (E) is the characteristic row. To obtain the characteristic row we make use of the modal column first to give us a first guess namely in the form:

$$E = \left(\frac{m_1 y_{11}}{m_m y_{1m}} \quad \frac{m_2 y_{12}}{m_m y_{1m}} \quad \frac{m_3 y_{13}}{m_m y_{1m}} \cdots \frac{m_n y_{1n}}{m_m y_{1m}} \right) \tag{19}$$

The above equation is normalized, that is, one of the terms is made equal to unity. With the normalized equation one iterates similarly to the previous way except that we have a row instead of a column. Once the solution has converged the following procedure is used.

We seek a new matrix (B) which will contain as its fundamental frequency, the second mode frequency. We must therefore sweep out the first mode frequency. We do this as is shown below. Let

$$B_1 = (B)(I-E) \tag{20}$$

where (I) is an identity matrix of the form shown in eq. (21). This matrix has as many rows as it has columns and has as many rows as matrix (B).

$$I = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix} \tag{21}$$

We then subtract the matrix (E) from (I). This must be done along one row only, the chosen row being the one corresponding to the column number used to normalize (E). Thus if the third term in the row of (E) is used to normalize, we subtract (E) from (I) in the third row. Once we have (I - E) we multiply it into (B). Since (I-E) usually has more than one column, the result will have more than one column. The method multiplication is as before but each column gives a column in the resultant matrix. When (B) is obtained, the same procedure is applied as we applied before to find the fundamental frequency, in other words

$$\frac{1}{\omega^2}(A) = (B_1)(A) \tag{22}$$

If we now apply this to our numerical example, we obtain the following: For our initial try for (E) we write

$$(E) = \left(\frac{[2.217][100/g]}{200/g} \quad \frac{[1.00][200/g]}{200/g}\right) = (1.1085 \quad 1.00)$$

where we use the deflections we found before. Next iterate by pre-multiplication to find out if the terms used are exact. Thus,

$$(1.1085 \quad 1.00)\begin{pmatrix}11.11 & 9.6\\ 4.8 & 4.8\end{pmatrix} = (17.1154 \quad 15.4416)$$

$$= 15.4416(1.1084 \quad 1.00)$$

$$(1.1085 \quad 1.00)\begin{pmatrix}11.11 & 9.6\\ 4.8 & 4.8\end{pmatrix} = (17.1143 \quad 15.44064)$$

The final result is

$$(E) = (1.1084 \quad 1.00) \tag{23}$$

Subtract the matrix (E) from (I). (I) needs to have only two rows since the matrix (B) only has two rows. Or,

$$(I) = \begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix}$$

so that

$$(I-E) = \begin{pmatrix}1 & 0\\ -1.1084 & 0\end{pmatrix} \tag{24}$$

the subtraction taking place along the second row since we used the second term to normalize (E). Then,

$$(B_1) = \begin{pmatrix}11.11 & 9.6\\ 4.8 & 4.8\end{pmatrix}\begin{pmatrix}1 & 0\\ -1.1084 & 0\end{pmatrix} = \begin{pmatrix}0.46936 & 0\\ -0.52032 & 0\end{pmatrix} \tag{25}$$

Thus the final matrix equation that we need to solve is

$$\lambda^2(A) = \begin{pmatrix}0.46936 & 0\\ -0.52032 & 0\end{pmatrix} \tag{26}$$

When the beam vibrates in the second modal shape, we know that one of the deflections is up and the other one is down. For simplicity let us start with unity deflections at both points

except that one is plus and the other is negative. Arranging our work again in tabular form as before we obtain the following:

TABLE II

	1	2	3	4	5	6	7
	(A)	$(\lambda^2 A)$	(A)	$(\lambda^2 A)$	(A)	$(\lambda^2 A)$	(A)
y_1	+1	0.46936	.90206	.42339	.46935	.220289	.46935
y_2	-1	-.52032	-1.0	-.90206	-1.0	-.46935	-1.0
			.52032		.90206		.46935

Thus the critical frequency is:

$$\lambda^2 = \frac{10^4 g}{\omega^2} = 0.46935 \; ; \quad \omega = 27{,}350 \text{ c.p.m.}$$

and the Prohl method gives the same result.

Summary:

We see here that the fundamental procedure is quite simple and straightforward. Any mistakes, unless made consistently, are rectified by subsequent iterations. By use of the calculating machine the amount of paper work is reduced to a minimum. The method removes one of the difficulties presented by the other methods, namely, the guessing of the frequencies. The objection to guessing the frequency is that we never really know to which frequency our results are converging, whereas here, though we must guess the deflections, convergence to the lowest frequency is quite rapid.

We may therefore summarize the method by stating that the matrix method first redefines the multiplication system. By so defining the multiplication system it is possible to show that the mathematical form of the problem is the well known characteristic value problem. Then by assuming the deflections and by successive steps or cycles of the same procedure one eventually finds that the deflections no longer change and the critical frequency can be extracted. In order to find the higher mode frequencies it is necessary to remove the influence of the lower modes and this is done in the manner shown in the paper. Once the reduced dynamic matrix is found the same procedure applies to finding the critical frequency of the higher mode than applied to finding the fundamental frequency.

The method can be extended to systems of different form and loading and only slight modifications are needed. The main advantage of the method is that it is rapid and the numerical work can be done by a computer.

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FREQUENCY RESPONSE ANALYSIS OF FEEDBACK CONTROL SYSTEMS

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1. INTRODUCTION

As is well known, the only really complete and conclusive method of analyzing a feedback control system or, in fact, any other type of dynamic system is to solve the differential equations for the system. This is possible for only a very few specific cases other than those in which the equations are linear and have constant coefficients. However, in these cases the frequency response method of analysis is also applicable. For this reason, frequency analysis response is more valuable than is often appreciated; namely, it gives in rather simple form a large portion of the information which could be had by other means in substantially all cases where an analytical solution was possible in the first place.

2. FREQUENCY RESPONSE PLOTS

Basically, frequency response analysis consists of considering the responses of the various components to a steady state sine wave excitation over a considerable range of frequencies encompassing two or three octaves on either side of the frequencies significant to the operation of the feedback control system.

There are two principal methods for plotting frequency response: the amplitude-versus-phase plot, sometimes loosely referred to as a Nyquist diagram, and the plot of amplitude and phase as functions of frequency, usually referred to as a log frequency plot. In the latter case, the logarithm of the magnitude of response is plotted against the logarithm of the frequency and the phase is plotted on an arithmetic scale against the logarithm of the frequency. Amplitudes are frequently expressed in decibels and frequencies in radians per second.

2.1 Ideal Motor

Figure 1 (solid curve) shows an amplitude-phase plot for an ideal motor in which the phase and amplitude of the motor speed are being plotted against each other on a polar diagram for the complete range of frequencies from minus infinity to plus infinity. It will be noted that the plot is a circle tangent to the imaginary axis at the origin and having its center on the real axis. This same characteristic is plotted in Figure 2 (solid line) on a log frequency scale. Here it will be noted that the speed response approaches unity at a phase angle of zero degrees for very low frequencies, while at high frequencies the response approaches an asymptotic straight line having a slope of six decibels per octave, or 20 decibels per decade, and that the phase shift approaches asymptotically to -90^0 or $-\frac{\pi}{2}$. Both of these facts are evident in a general way from the Nyquist plot. However, the log frequency plot brings out one or two other points. In particular, the response in amplitude is 3 db down at the so-called corner frequency, the frequency at which the 20 db per decade asymptote intersects the horizontal asymptote. At this point the phase shift is -45^0. Also, the phase shift graph is anti-symmetric with respect to the corner frequency and the value -45^0. These phase and amplitude plots are characteristic of any type of linear circuit having

a simple lag such as, for example, a coil and resistor in series. Since we are plotting speed rather than displacement, it is evident that a motor is equal to a coil and resistor with an integrator attached.

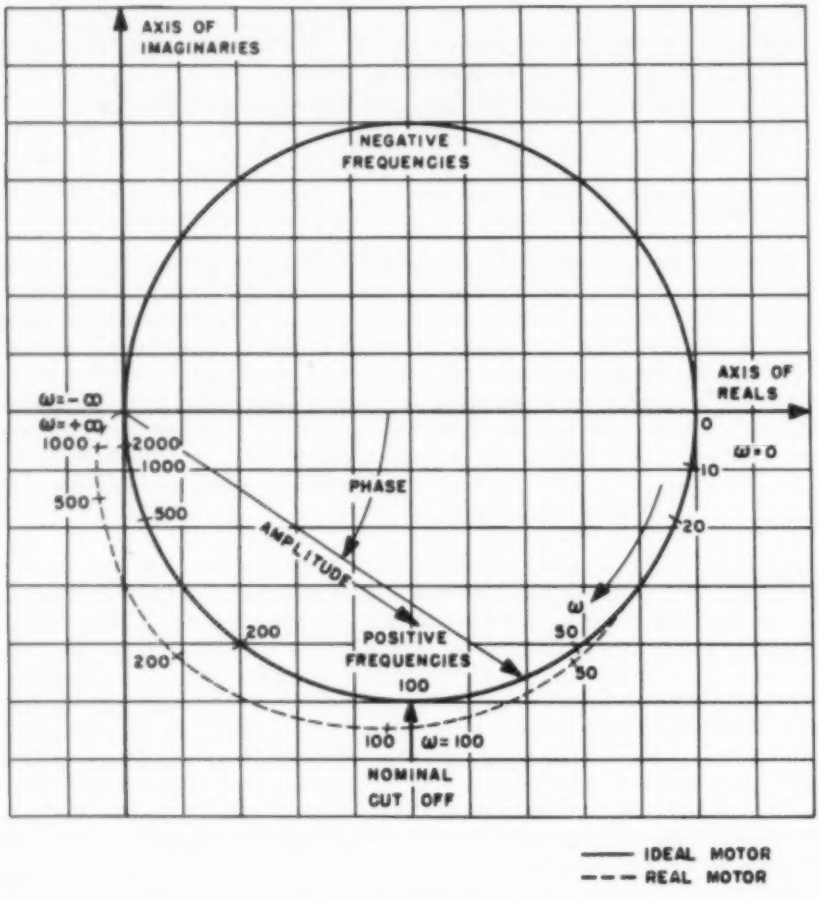


Figure 1. Amplitude vs. Phase of Motor Speed

2.2 Real Motor

Actually, the majority of real motors do not perform quite like this ideal, particularly at the higher frequencies. The dotted lines on Figures 1 and 2 indicate the type of deviations usually experienced. It will be noted that the log frequency plot for this physical motor may be closely approximated by a series of three straight lines having slopes 0, 20 db per decade, and 40 db per decade. It may be further noted that at the second corner frequency the phase shift is very nearly equal to -135°. The performance of a real motor is therefore very similar to the performance of an RLC circuit somewhat more than critically damped and containing, in addition, an integrator.

3. MINIMUM PHASE NETWORKS - BODE'S THEOREM

One of the most significant points about the phase-amplitude plots is the fact that for a large class of networks — referred to as minimum phase networks — the phase characteristics can be specified when the amplitude characteristic is known, or vice versa. For

purposes of this discussion, a simple definition of a minimum phase network is one in which there is only one path for signals to reach the output from the input. For example, a π section as indicated in Figure 3 is a minimum phase network, whereas the bridged T and lattice shown in Figures 4 and 5 are not minimum phase networks. The term "minimum phase" refers to the fact that networks of this class have the smallest possible amount of phase shift which can be associated with the specified amplitude change in any physically realizable linear dynamic system. The relations between amplitude and phase in a minimum phase network are expressed by Bode's phase amplitude theorem

$$B_c = \frac{\pi}{12}\left(\frac{dA}{du}\right)_c + \frac{1}{6\pi}\int_{-\infty}^{\infty}\left[\frac{dA}{du} - \left(\frac{dA}{du}\right)_c\right] \ln \coth \frac{u}{2}\, du$$

where B_c = phase shift, radians, dA = attenuation slope db/octave, u ln f/fc. Bode's equation states that the phase shift at any frequency is equal to 90° for each 6 db per octave slope posessed by the log amplitude plot plus a correction indicating the effect which portions of the attenuation curve of different slopes at remotely located frequencies may have on the phase shift at the point under consideration. In particular, the weight function expressed in the integral is given by the graph in Figure 6 (Note that this weight function may also be expressed as $\ln \left(\frac{f + fc}{f - fc}\right)$. This weight function indicates that portions of the attenuation curve which are removed one octave or more have only moderate effect on the phase, and that portions one decade or more away from the point under consideration have essentially no effect.

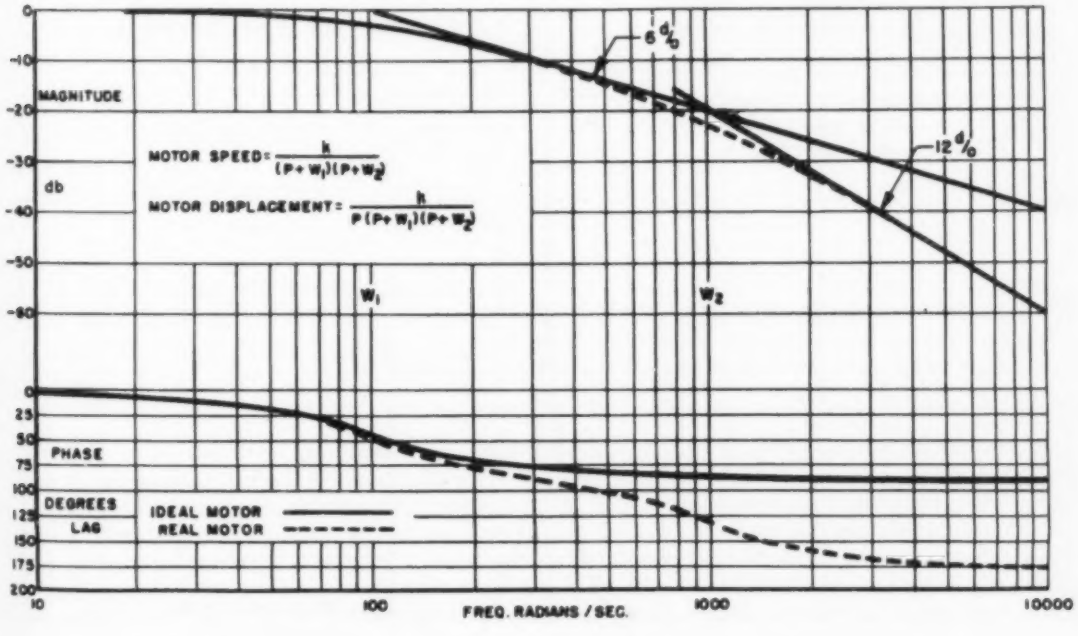


Figure 2. Phase 8 Magnitude of Speed vs. Log Frequency

3.1 Standard Phase Chart

Because of the direct relationship between amplitude and phase, it is therefore possible to make up a permanent chart for the phase corresponding to a simple standard amplitude pattern and break up all amplitude curves into combinations of the simple

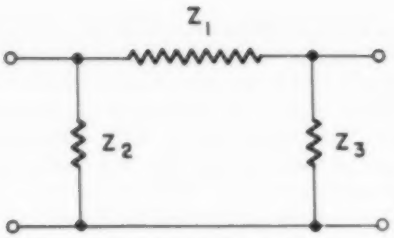


Figure 3. A "π" Section
(Minimum Phase Network)

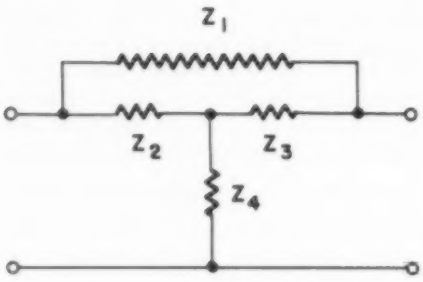


Figure 4. A Bridged "T" Section
(Non Minimum Phase Network)

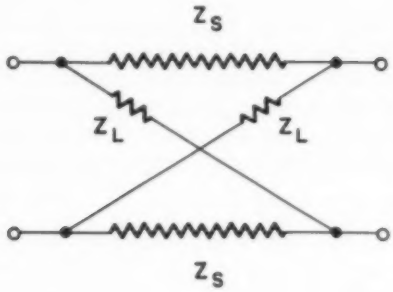
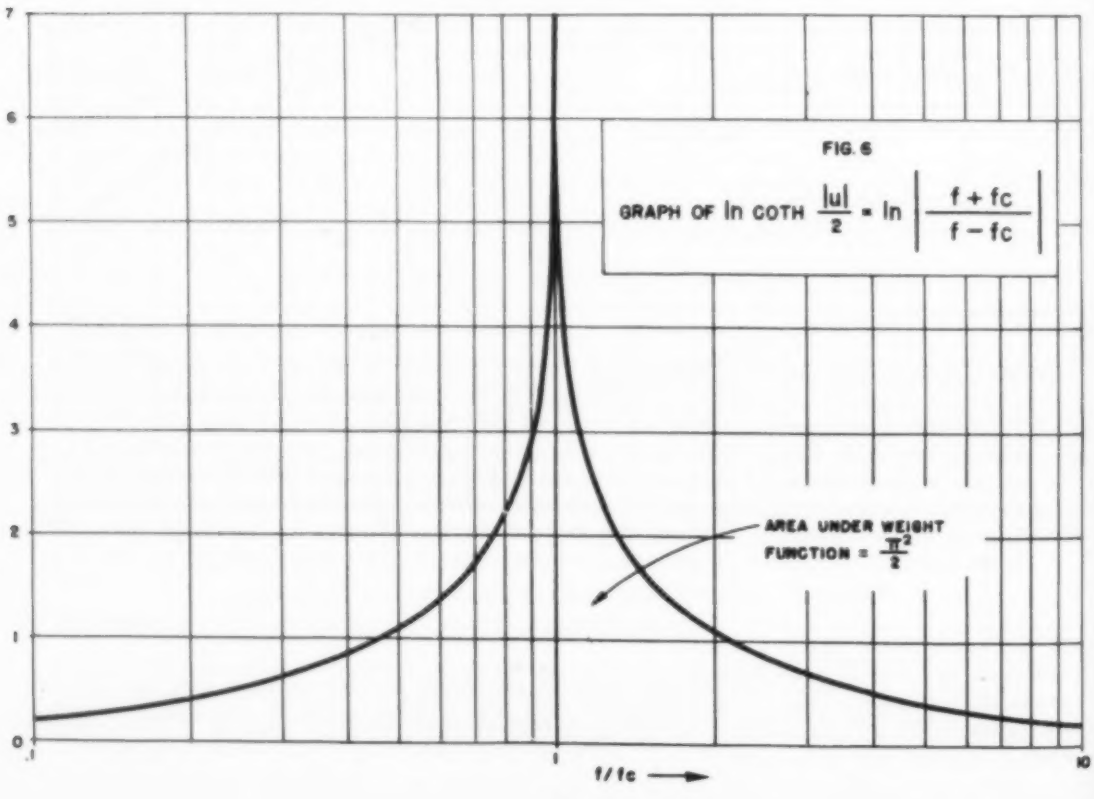


Figure 5. A Lattice Section
(Non Minimum Phase Network)



curve with different slopes and corner frequencies. This is of great value in stabilizing feedback control systems in which characteristics of the components must be determined experimentally. It is often found that determination of amplitude response is comparatively simple and straightforward, while correct measurements of phase are rather difficult to make in many cases. The simple phase characteristic from which all others can be calculated is shown in Figure 7. In general, it is sufficiently

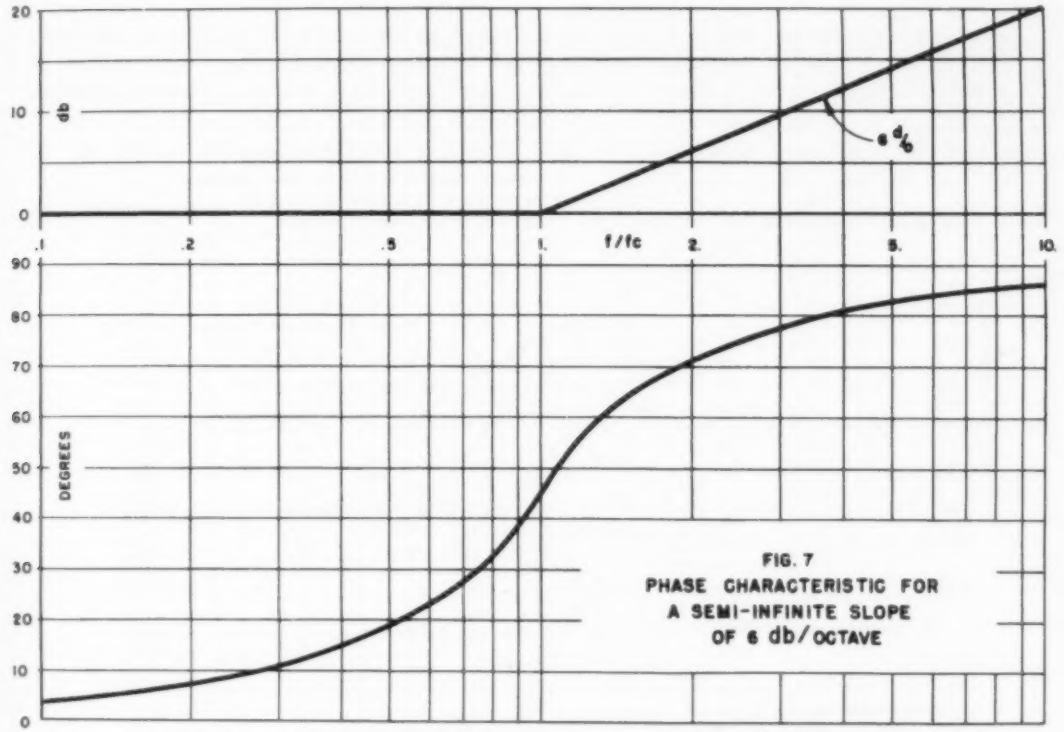


FIG. 7
PHASE CHARACTERISTIC FOR
A SEMI-INFINITE SLOPE
OF 6 db/OCTAVE

accurate to approximate the given amplitude characteristic by a series of straight lines having slopes which are multiples of 6 db per octave, and determine the corresponding phase angles from the graph shown in Figure 7. Phases on one side only of the corner frequency need to be plotted since they are anti-symmetric with respect to the corner frequency.

We may now draw amplitude and phase characteristics for the several components of a feedback control system on a log frequency plot and add the results arithmetically in order to give the over-all frequency and phase responses for the network. We may then, if we wish, draw the Nyquist type plot for the same circuit and determine from an examination of this plot what degree of stability is possessed by the network and the values of the phase and amplitude margins. These quantities are incidentally also available directly on the log frequency plot in addition to the graphical estimate of the velocity constant and the acceleration constant at least for simple cases.

4. USE OF THE FREQUENCY RESPONSE METHOD

The frequency response method of analysis for servomechanisms is most useful where it is difficult or impossible to determine the actual analytic values for the performance of the components and it is desired to determine in a rough fashion what specific changes need to be made in the components in order to imporve the degree of stability, velocity

response, etc. It is this particular point in which the frequency response method of analysis excels over all others. An examination of the log frequency plot, or of the Nyquist diagram, makes it immediately evident what modifications are required in order to alter the characteristics of the loop in the required direction. It is also possible to estimate with reasonable accuracy the amount of modification required.

4.1 Example of Frequency Response Analysis

Perhaps the simplest way to explain the operation of the method is to take an example, Let us assume that a motor having the characteristics shown in Figure 2 replotted in terms of displacement in Figure 8A is to be used to drive a servomechanism in which

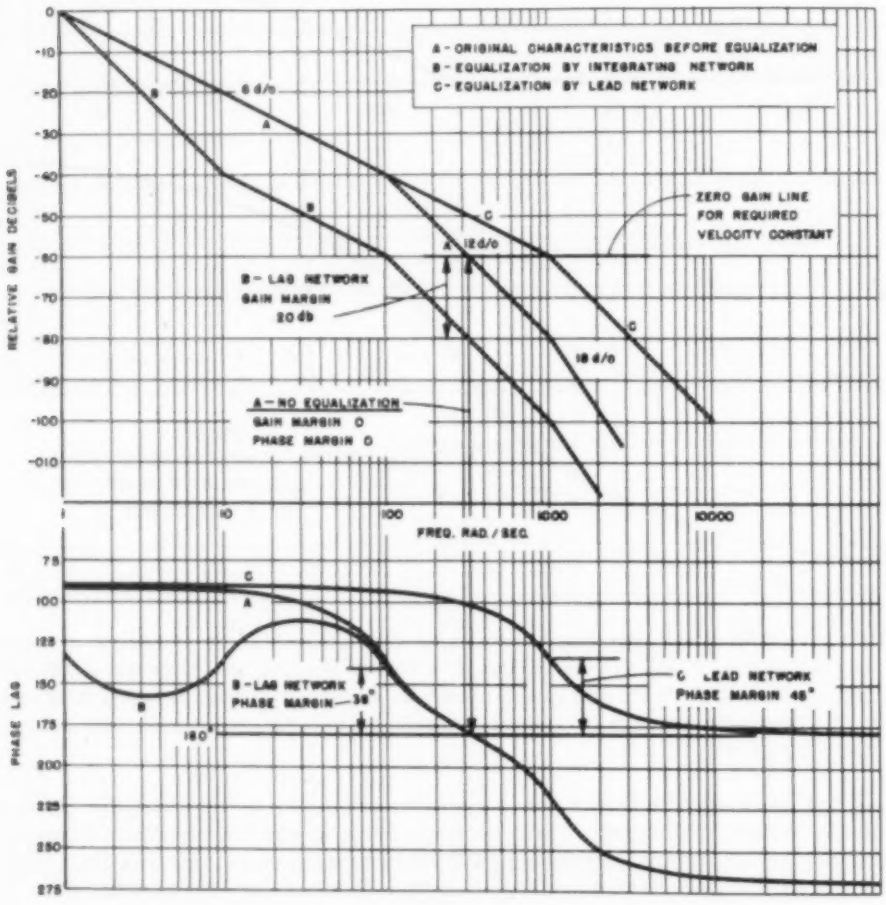

A - ORIGINAL CHARACTERISTICS BEFORE EQUALIZATION
B - EQUALIZATION BY INTEGRATING NETWORK
C - EQUALIZATION BY LEAD NETWORK
RELATIVE GAIN DECIBELS
6 d/o
12 d/o
18 d/o
ZERO GAIN LINE FOR REQUIRED VELOCITY CONSTANT
B - LAG NETWORK GAIN MARGIN 20 db
A - NO EQUALIZATION GAIN MARGIN 0 PHASE MARGIN 0
FREQ. RAD./SEC.
PHASE LAG
B - LAG NETWORK PHASE MARGIN - 38°
C LEAD NETWORK PHASE MARGIN 48°
180°

Figure 8. Log Frequency Plot for Motor Displacement vs. Frequency

the requirements are that the steady state error in response to a constant velocity input shall be not more than one part 1000; that is to say, there must not be an error of more than one radian when the input is calling for a constant velocity of 1000 radians per second. This velocity constant implies that at a frequency of one radian per second the open loop gain of the servo must be at least 1000 which is 60 db, or conversely that the gain must not fall below unity out to a frequency of 1000 radians per second (provided that the attenuation slope is 6 db per octave throughout). When the servo amplifier gain is adjusted to the point where this condition is satisfied, it is evident

from the log frequency diagram that the loop will not be stable inasmuch as the phase shift will be 180° when the gain of the open loop has become equal to unity. This may be seen in another way by examining the Nyquist diagram corresponding to this situation. The graph passes through the critical point which indicates there will be a sustained oscillation.

4.2 Stabilization by Integrating Network

Generally speaking, there are two ways in which the servo can be stabilized and caused to perform in the desired fashion. The first of these is to introduce an equalizer or corrective network which causes the gain to drop from the value of 1000 required at one radian per second down to a value near zero before frequencies in the vicinity of the cutoff frequency of the motor have been reached. This procedure will introduce

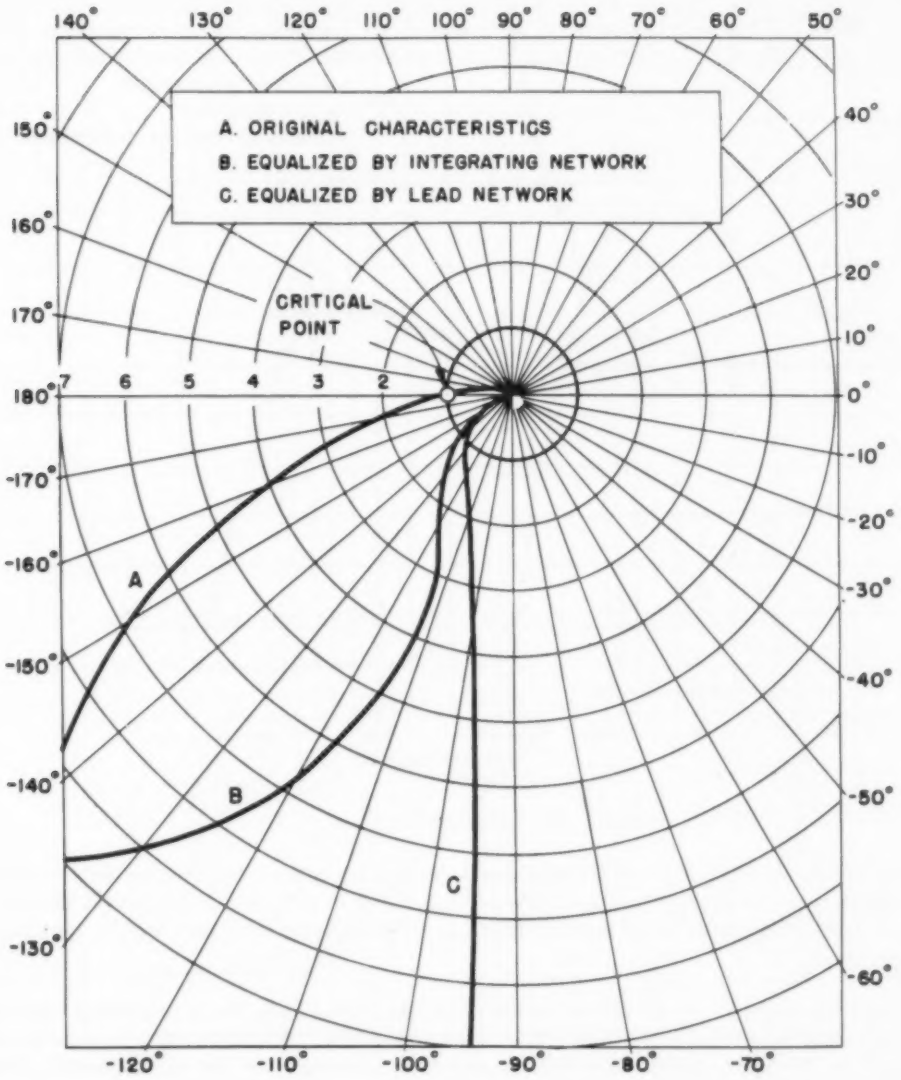


Figure 9. Nyquist Diagram

additional phase shift at the lower frequencies, but after the attenuation characteristic has been leveled out after the sharp drop the phase will return to its former value. It may be stated in general that with the gain dropping off at not more than 8 or 10 db per octave maximum in the vicinity of cutoff, it will be possible to pass the unity gain point without incurring large phase shifts. The characteristics of the servo so modified also appear in Figure 8B.

4.3 Stabilization by Lead Network

Another comparatively obvious way in which the performance of the servo can be made to meet the requirements is to forestall the approach of high phase shift in the vicinity of motor cutoff so as to permit a 6 db per octave attenuation slope to exist from the

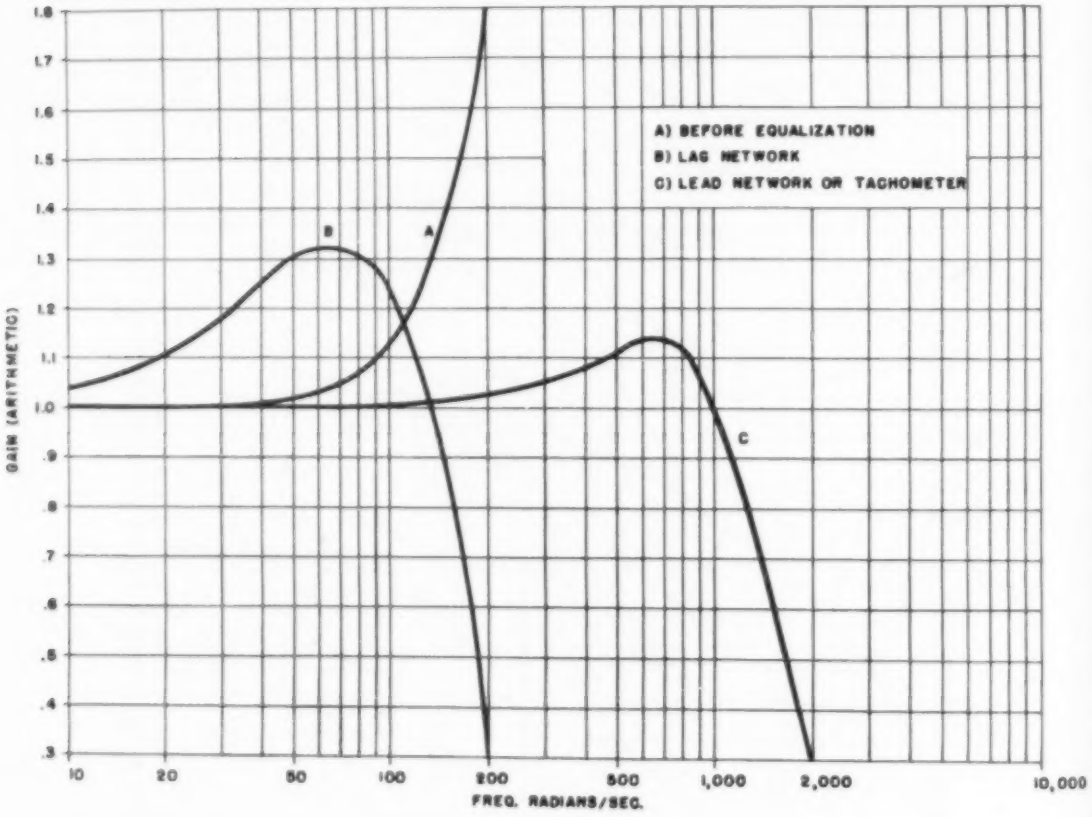


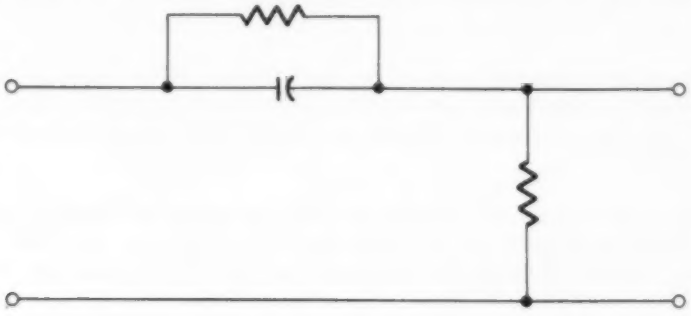
Figure 10. Closed Loop Gain

lowest frequencies out to 1000 radians per second or more. There are two techniques by which this can easily be accomplished and which to a first order approximation are equivalent. The first involves introducing a phase lead into the loop by means of a tachometer.

As will be noted from Bode's equation, introducing a phase lead is exactly the same as introducing a rising characteristic with frequency. A tachometer generator connected to the motor shaft will give an output proportional to the motor speed. If now this tachometer signal is introduced into the backward portion of a minor loop enclosing the motor and amplifier, the characteristic of the motor can be made to appear to

be 6 d/o out to approximately the region of the second corner frequency of the motor, or about 1000 radians per second. It will be noted that the tachometer rising characteristic is only desired in the region from 100 radians per second upward. Therefore, the tachometer output will be introduced to the amplifier through a high pass filter involving essentially a resistor and a capacitor. The over-all characteristic of the servo then appears as in Figure 8C.

The second method for correcting the performance of the servo whose motor has an inadequate bandwidth is to introduce in series with the forward portion of the main loop a section having a rising frequency characteristic, in other words, a differentiating circuit. Such a circuit is shown together with its log frequency characteristic in Figure 11. It should be noted that the characteristic of the differentiating circuit cannot rise indefinitely because of the effect of stray capacitances, etc. Also, this would require infinite gain in the amplifier. The differentiating circuit introduces a large loss at low frequencies and zero loss at points beyond the upper knee. The same servomechanism may be stabilized by means of a lead network in the forward loop resulting in an over-all characteristic similar to that shown in Figure 8C.



LEAD NETWORK

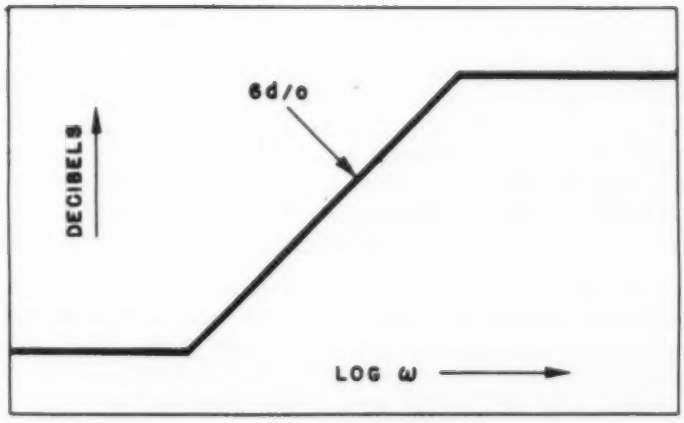


Figure 11. Amplitude Response

SIMPLIFICATION OF CONTACT GEOMETRY FOR SPIRAL GEAR LOAD CALCULATION

WERNER F. VOGEL, Assoc. Prof. of Engineering Mechanics, Wayne University, Detroit and
DOUGLAS HUGHSON, Transmission Project Engineer, Packard Motor Co., Detroit.

Heinrich Hertz established the general theory of stress and strain on and in solids, whose curved surfaces are pressed against each other by equal and opposite forces acting in the direction of the common normal, see Ref. 2.

Special solutions were developed by his followers, who are listed in the bibliographies of Ref. 1 and 3. The most comprehensive and convenient summary of such solutions may be found in Ref. 3.

These equations can be applied as soon as the two principal curvatures of each of the pressed surfaces are known in magnitude and relative position to each other. The latter can be expressed by one angle which, for the sake of distinction and simplicity, we will denote as the Hertzian Angle "H".

For crossed-axes helical gearing[1] the determination of these geometrical data was given for the first time by Victor Francis in his paper before this Society. (See Ref. 1). His and our investigation deal exclusively with the preponderant type having involute helicoids as tooth surfaces.

He started with the concept of two such involute helicoids contacting each other only at one common point of a pair of their straight generatices. The existence of straight generatices implies the simplification that the minimum curvature of each of these surfaces is zero.

He then established simple and exact equations for the maximum curvatures. This part of his derivation does not need any improvement.

Simplified Contact Geometry

However his procedure of derivation of the Hertzian Angle H was quite cumbersome and, consequently, his resulting equations are unnecessarily involved. A considerable simplification in the way of derivation and in the structure of the final equations will be shown here by starting all derivations from the kinematic concept of the common "conjugate helical rack" rather than from that of the involute helicoids. This preferable concept leads to simpler geometry, because it replaces the consideration of the curved surfaces of the helicoids by that of the plane surfaces of the rack. This improvement yields extremely simple projective relations.

The common helical rack should be understood as an imaginary rack, whose surfaces are shims placed between the contacting surfaces of the spiral gears. Fig. 1 shows such a helical rack and its relative position to one of the gears. It also illustrates the ribbon of contact, representing the "Plane of Action" of this gear. This ribbon is tangential to the base cylinder of the gear and normal to the surface of the rack tooth.

[1] "Crossed-Axes Helical Gears" are frequently designated as "Cylindrical Spiral Gears" or simply as "Spiral Gears".

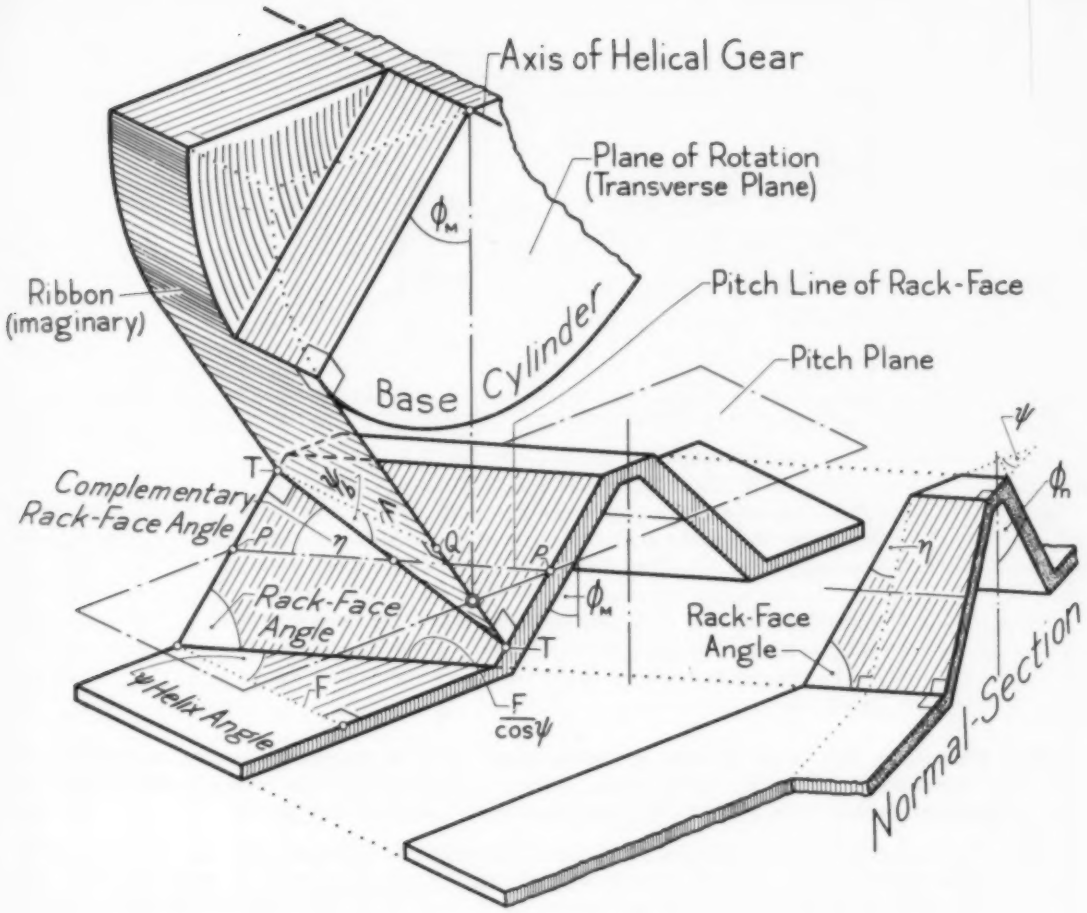


Figure 1. MODEL OF HELICAL SHIM RACK

Concept: The imaginary shim rack (of infinitesimal shim thickness) is inserted between two meshing (not shown) helical involute teeth. Each of these teeth, (if colored), will print its contacting generatrix on the rack-face, (e.g. line T -- T generatrix of upper gear tooth; generatrix of lower gear tooth not shown here, but in Fig. 2). T -- T also is the trace of the ribbon of contact (≡ "plane of action" of the upper gear) on the rack-face. The "Rack-Face Angle" and its complement "η" are newly introduced terms with important applications, (See Table 1). For other angular notation and relations see Table 2.

If two contacting crossed-axes involute gears are under consideration their two ribbons intersect each other in a straight line, (projected in Point "N" of Fig. 2), which is the locus of all consecutive contact points between the gear teeth. This line is usually labelled as their path of contact. It pierces perpendicularly the contacting surfaces of the teeth and their imaginary rack.

The significance of these geometric facts for the application of Hertz' Theory can be stated in the following propositions:

1. The intersecting line between the two ribbons of contact is the common normal of the contacting tooth surfaces. Therefore, this line is the "line of action" of the tooth load, if friction is disregarded. (See line "N", perpendicular to the rack-face in Fig. 2).
2. The constant angle between the two ribbons of contact is the (required) Hertzian Angle. This proves that the Hertzian Angle is not influenced by the movement of the teeth to a different contact position.

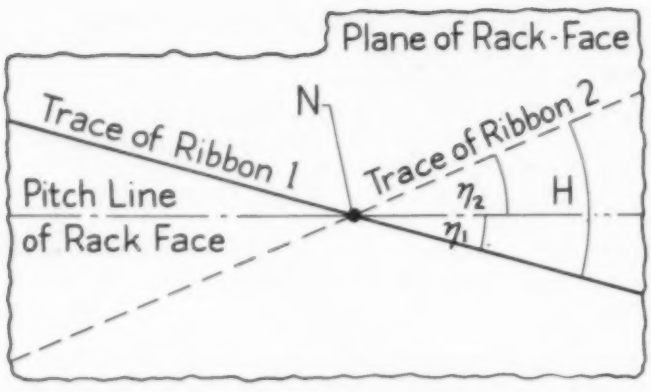


Figure 2. HERTZIAN ANGLE H
FOR CROSSED-AXES HELICAL GEARS

The ribbons of both gears are perpendicular to the rack-face and intersect each other in the common rack-face normal, (projected in point "N"). The Hertzian Angle "H" is the angle between the two ribbons. For calculation see Table 1. The intersecting traces of the ribbons may also be interpreted as the pair of instantaneously contacting generatrices of the (not shown) meshing spiral gear teeth.

3. Therefore, this angle always presents itself in true magnitude on the surface of the helical rack tooth. It is the angle formed by the two traces in which the ribbons of contact pierce this rack tooth. (See traces "1" and "2" in Fig. 2.)

4. The traces of (1) and (2), described in Proposition 3, coincide with the two contacting generatrices of the meshing involute helicoids. This corroborates Proposition 2 and our claim that the angle between the traces "1" and "2" is the required Hertzian Angle.

5. According to Fig. 2 and Table 1, the Hertzian Angle can always be easily obtained as either the sum or the difference of the (newly introduced) "Complementary Rack-Face Angles", η_1 and η_2, of the two meshing gears.

6. The "Complementary Rack-Face Angle, η", of any Involute Gear may be established from any of the simple equations derived in the next chapter and listed in the fifth column of Table 2. The selection of the most suitable equation depends on the given specifications of the gears.

In crossed-axes helical gearing the different specifications of Gears 1 and 2 will result in two different numerical values of η_1 and η_2, respectively.

TABLE 1 · THE HERTZIAN ANGLE FOR INVOLUTE GEARS

The subscripts "1" and "2" refer to the meshing gears, respectively.
The Hertzian Angle is independent from their instantaneous contact position.

TYPE OF GEARS	HANDS OF MESHING TEETH	SHAFT ANGLE	HERTZIAN ANGLE
CROSSED-AXES HELICAL GEARS	Equal	$\Sigma = \psi_1 + \psi_2$	$H = \eta_1 + \eta_2$
	Opposite	$\Sigma = \psi_1 - \psi_2$	$H = \eta_1 - \eta_2$
PARALLEL-AXES HELICAL GEARS $\psi_1 = \psi_2$; $\eta_1 = \eta_2$	Opposite	$\Sigma = 0$	$H = \eta_1 - \eta_2 = 0$
SPUR GEARS $\psi_1 = \psi_2 = 0$; $\eta_1 = \eta_2 = 0$	Neutral	$\Sigma = 0$	$H = 0 \pm 0 = 0$

TABLES 2-5. RELATIONS BETWEEN THE SIGNIFICANT ANGLES OF A HELICAL INVOLUTE GEAR

Only three of the six significant angles need to be given to determine the others.

FUNCTION	PRESSURE ANGLES		HELIX ANGLES		COMPLEMENTARY
	Main Pressure Angle (in plane of rotation)	Normal Pressure Angle (in normal plane)	(Pitch) Helix Angle	Base Helix Angle	RACK-FACE ANGLE (in face of helical rack)
Sine	$\sin\phi_M = \frac{\sin\phi_n}{\cos\psi_b} = \frac{\sin\eta}{\sin\psi}$	$\sin\phi_n = \sin\phi_M \cdot \cos\psi_b = \frac{\tan\eta}{\tan\psi}$	$\sin\psi = \frac{\sin\psi_b}{\cos\phi_n} = \frac{\sin\eta}{\sin\phi_M}$	$\sin\psi_b = \sin\psi \cdot \cos\phi_n = \frac{\tan\eta}{\tan\phi_M}$	$\sin\eta = \sin\phi_M \cdot \sin\psi = \tan\phi_n \cdot \tan\psi_b$
Cosine	$\cos\phi_M = \frac{\tan\psi_b}{\tan\psi} = \cos\eta \cdot \cos\phi_n$	$\cos\phi_n = \frac{\sin\psi_b}{\sin\psi} = \frac{\cos\phi_M}{\cos\eta}$	$\cos\psi = \frac{\tan\phi_n}{\tan\phi_M} = \cos\eta \cdot \cos\psi_b$	$\cos\psi_b = \frac{\sin\phi_n}{\sin\phi_M} = \frac{\cos\psi}{\cos\eta}$	$\cos\eta = \frac{\cos\phi_M}{\cos\phi_n} = \frac{\cos\psi}{\cos\psi_b}$
Tangent	$\tan\phi_M = \frac{\tan\phi_n}{\cos\psi} = \frac{\tan\eta}{\sin\psi_b}$	$\tan\phi_n = \tan\phi_M \cdot \cos\psi = \frac{\sin\eta}{\tan\psi_b}$	$\tan\psi = \frac{\tan\psi_b}{\cos\phi_M} = \frac{\tan\eta}{\sin\phi_n}$	$\tan\psi_b = \tan\psi \cdot \cos\phi_M = \frac{\sin\eta}{\tan\phi_n}$	$\tan\eta = \tan\phi_M \cdot \sin\psi_b = \sin\phi_n \cdot \tan\psi$

$\frac{\sin 2\phi_M}{\sin 2\phi_n} = \frac{\cos\eta}{\cos\psi_b}$	$\frac{\sin 2\phi_M}{\sin 2\eta} = \frac{\cos\phi_n}{\sin\psi}$	$\frac{\sin 2\phi_n}{\sin 2\psi_b} = \frac{\sin\phi_M}{\sin\psi}$	$\frac{\sin 2\psi}{\sin 2\psi_b} = \frac{\cos\eta}{\cos\phi_n}$	$\frac{\sin 2\psi}{\sin 2\eta} = \frac{\cos\psi_b}{\sin\phi_M}$

$$\frac{1+\cos 2\phi_M}{1+\cos 2\phi_n} = \frac{1+\cos 2\psi}{1+\cos 2\psi_b} = \frac{1+\cos 2\eta}{2}$$

Axial Pressure Angle ϕ_x	$\tan\phi_x = \frac{\tan\phi_M}{\tan\psi} = \frac{\tan\eta}{\tan\psi \cdot \sin\psi_b} = \frac{\sin\phi_n}{\sin\psi_b}$

For the actual calculation of helical gear load, the authors suggest to use the procedure outlined in Francis' paper, Ref. 1, in which only the chapter on the Hertzian Angle "H" should be replaced by the described concept of our illustrations and the less involved equations of our Tables 1 and 2.

The Complementary Rack-Face Angle

Fig. 1 will now be used to derive a general equation for the angle "η" between the pitch line (P .. P) of the rack-face and the trace (T .. T), in which the ribbon of contact penetrates the rack-face.

We can conclude from the rectangular triangle QTT (on the ribbon), that

$$\overline{TT} = \frac{F}{\cos \psi_b} \tag{1}$$

The letter "F" used in this equation and in Fig. 1 is the standardized symbol for the "Face Width", as defined by the American Standard "Letter Symbols for Gear Engineering."

Furthermore, the length P .. P of the pitch line of the rack-face is the upper side of a parallelogram and thus can be read as:

$$\overline{PP} = \frac{F}{\cos \psi} \tag{2}$$

Since it can be proved that the angle PTT equals 90°, we also can write:

$$\overline{PP} = \frac{\overline{TT}}{\cos \eta} \tag{3}$$

Then, the substitution of equations (1) and (2) into (3) yields :

$$\boxed{\cos \eta = \frac{\cos \psi}{\cos \psi_b}} \tag{4}$$

The authors find it useful to introduce the name "Rack-Face Angle" for the acute angle of the parallelogram, which represents the rack-face in the left-hand part of Fig. 1. It becomes then apparent that the angle "η", derived from the kinematic concept of the "ribbon of contact," has the geometric significance of "THE COMPLEMENTARY RACK-FACE ANGLE".

The complementary relationship between the two angles may also be read from the triangle in the right-hand part of Fig. 1. This part of the illustration also shows the "Normal Section" of the shim rack, from which, by simple projective relations, one may easily derive[2] the equation :

$$\boxed{\cos \eta = \frac{\cos \phi_M}{\cos \phi_n}} \tag{5}$$

[2] Credit for this direct derivation of Equation (5) is gratefully extended to Mr. John Slater of the Ford Motor Co., who submitted it to Dr. Vogel as early as 1945. At that time the usefulness of this equation was not fully recognized.

However, it is unnecessary to reproduce this derivation because the identity of the fractions of Equations (4) and (5) is a well established fact. This fact and, in addition, a complete system of the relations between the pressure and helix angles of any helical involute gear is stated in Table 32d of Vogel's book: Involutometry and Trigonometry, (See Ref. 1).

The utilization of some of these relations made it immediately possible to write a complete set of equations for all trigonometric functions of the new "Complementary Rack-Face Angle," (See column 5 of Table 2). The doubly framed expression of this column offers now a geometric interpretation of the previously known fractional identities of the shown cosine functions.

The recognition of the "Complementary Rack-Face Angle" as a new significant value of any involute screw made it advisable to extend the quoted Vogel-Table to a system of relations between all "significant angles" of a helical gear (or any other involute screw). These exact equations are shown in Tables 2 to 5.

Their usefulness is not confined to gear load calculations, but the rather general importance lies in their universal geometric validity and applicability.

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